# Research Statement

The study of homotopy theory originates in algebraic topology, with the investigation of homotopy invariants such as (co)homology groups or homotopy groups. By passing from topological spaces to spectra, one can further study *stable* invariants, and in the presence of a group action, one can study *equivariant* invariants as well. Equivariant stable homotopy theory has proven widely useful in recent years, with applications in showing the the non-existence of elements of Kervaire invariant one [HHR16]; in computing algebraic K-theory and variants [NS17] [BMS19]; and in studying chromatic homotopy theory, derived algebraic geometry, and representation theory.

In my research, I apply the computational methods of homotopy theory to answer questions about the **modular representation theory** of finite groups $G$ over a field $k$ of characteristic $p$, where $p$ divides the order of $G$. Modular representation theory was revolutionized by a focus on an invariant denoted StMod$(kG)$, the **stable module category of** $G$. Notably, StMod$(kG)$ has a homotopy-theoretic interpretation as a stable symmetric-monoidal $\infty$-category. Correspondingly, questions in representation theory can be turned into questions about homotopy invariants.

For example, one particular problem of interest is in computing the **group of endo-trivial modules** $T(G)$. That is, the $kG$-modules $M$ such that the endomorphism module decomposes as End$_k(M) \cong k \oplus P$, where $k$ is the trivial $kG$-module, and $P$ is a projective $kG$-module. Endo-trivial modules are ubiquitous in representation theory, for example in understanding simple modules for p-solvable groups, or in studying Morita or derived equivalences in block theory.

$T(G)$ has a homotopy-theoretic interpretation as the **Picard group** Pic(StMod$(kG)$), which consists of objects in StMod$(kG)$ that are invertible with respect to the tensor product. Picard groups are an interesting and tractable invariant, and were studied in stable homotopy theory first by Hopkins [HMS92]. In chromatic homotopy theory, the study of Picard groups is a deep and active field of research [HS99], [KS04], [GHMR12], [HMS17].

## Recent Results

In my thesis work, joint with van de Meer, I used homotopy-theoretic techniques to calculate Pic(StMod$(kG)$) for cyclic $p$-groups and (generalized) quaternion groups using the descent ideas and techniques of Mathew-Stojanoska [MS16]. Rather than study the Picard group Pic$(\mathcal{C})$, I instead study the **Picard *space*** $\mathcal{P}ic(\mathcal{C})$, which recovers Pic$(\mathcal{C}) \cong \pi_0(\mathcal{P}ic(\mathcal{C}))$. Moreover, I show that **Galois descent** holds for these groups, which allows me to use tools from equivariant stable homotopy theory, namely the **homotopy fixed point spectral sequence** (HFPSS), to calculate $\pi_*(\mathcal{P}ic(\text{StMod}(kG))$.

**Theorem A** (van de Meer-W. [vdMW21], cf Carlson-Thévenaz [CT00])**.** *Let $p$ be any prime, and let $n \geq 2$.*

$$\text{Pic}(\text{StMod}(kC_{p^n})) \cong C_2$$

**Theorem B** (van de Meer-W. [vdMW21], cf Carlson-Thévenaz [CT00])**.** *Let $p = 2$, and let $\omega$ denote a cube root of unity.*

$$\text{Pic}(\text{StMod}(kQ_{2^n})) \cong \begin{cases} C_4 & \text{if } n = 3 \text{ and } \omega \notin k \\ C_4 \oplus C_2 & \text{if } n = 3 \text{ and } \omega \in k \\ C_4 \oplus C_2 & \text{if } n \geq 4 \end{cases}$$

Notably, these proofs provide new insights into the classical work of Carlson-Thévenaz [CT05], who originally computed $T(G)$ by explicitly constricting endotrivial modules using group cohomology and the theory of support varieties. In particular, our proof provides a satisfying conceptual explanation for why the cube root of unity $\omega$ is significant in the case $G \cong Q_8$, but not for $Q_{2^n}$.

In more recent work, I have extended my methods to computations to the class of **groups with periodic cohomology**. Periodicity is a condition on the **group cohomology of** $G$: $H^*(G;k)$, and is equivalent to the statement that every abelian subgroup of $G$ is cyclic. These groups have been classified completely by work of Suzuki-Zassenhaus. Note that $C_{p^n}$ and $Q_{2^n}$ are precisely the $p$-groups with periodic cohomology. In particular, I extended my computations to the non $p$-group case, including metacylic groups (e.g. $S_3$ or $D_{2p}$), or the binary tetahedral group $T \cong Q_8 \rtimes C_3 \cong SL_2(\mathbb{F}_3)$.

**Theorem C** (W., cf Carlson-Mazza-Nakano)**.** *Let $S_3$ be the symmetric group on 3 elements.*

$$\mathrm{Pic}(\mathrm{StMod}(kS_3)) \cong \begin{cases} \{e\} & \textit{if } char(k) = 2 \\ C_4 & \textit{if } char(k) = 3 \end{cases}$$

Extending to non $p$-groups requires further representation-theoretic input (e.g. block theory and Brauer characters). Ultimately, I was able to show that Galois descent holds for this class of groups; and thus calculate $\pi_*(\mathcal{P}ic(\mathrm{StMod}(kG))$ using the homotopy fixed point spectral sequence.

## Future directions

I propose to extend my work in a few different directions.

1. One essential step in my work is in showing that StMod($kG$) was compactly generated by the trivial module. In homotopy-theoretic terms, there is an equivalence of symmetric monoidal $\infty$-categories [SS03] [Mat15b]
$$\mathrm{StMod}(kG) \cong \mathrm{Mod}(k^{tG})$$
where $k^{tG}$ is a commutative ring spectrum called the Tate construction of $G$. I propose to determine the precise class of groups $G$ for which we have such an equivalence.

2. The techniques I use are a special case of a broadly useful technique: there is a homotopy limit decomposition $\mathrm{StMod}(kG) \simeq \mathrm{holim}_{G/H \in \mathcal{O}_{\mathcal{A}}(G)^{op}} \mathrm{StMod}(kH)$ [Mat16], which reduces the difficult problem of understanding StMod($kG$) to understanding how simpler categories StMod($kH$) assemble together. For the groups I have studied, this homotopy limit is the homotopy fixed points, and so we are in the setting of Galois descent.

   However, for other groups (such as $D_8$), this limit is more difficult to study. Understanding the limit decomposition in this case would be the first example in understanding *descent with respect to elementary abelian subgroups*, which would be of independent interest in equivariant homotopy theory. I propose to study the $D_8$ case to better understand this homotopy limit decomposition.

3. Finally, I propose to use my methods to construct and compute novel homotopy invariants that have representation-theoretic significance. For example, in modular representation theory, one can consider the Dade group of **endo-permutation modules** for a $p$-group $P$. That is, the group of $kP$-modules $M$ where $\mathrm{End}_k(M) \cong k \oplus kY$, where $Y$ is a $P$-set [Dad78]. I propose to categorify the Dade group by constructing a Dade *space*, and studying this space via descent methods. This space would in turn provide new homotopy-theoretic invariants to study.

   I also propose to use these descent methods to compute the currently unknown Picard groups for variants of StMod($kG$) coming from tensor-triangular geometry [Bal10] [BIK08b]. This would produce new and interesting objects in modular representation theory.

# Background

## Picard Groups

Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a symmetric monoidal category. The **Picard group** of $\mathcal{C}$ is the group of isomorphism classes of objects $x \in \mathcal{C}$ such that $x$ is invertible with respect to the symmetric monoidal product $\otimes$ in $\mathcal{C}$. That is, there exists an object $y \in \mathcal{C}$ such that $x \otimes y \simeq \mathbb{1}$. For example, if $R$ is a PID, then $(\mathrm{Mod}(R), \otimes, R)$ is a symmetric monoidal category with $\mathrm{Pic}(\mathrm{Mod}(R))$ trivial. A more interesting example is the category of spectra $(\mathrm{Sp}, \wedge, \mathbb{S})$, which has $\mathrm{Pic}(\mathrm{Sp}) \cong \mathbb{Z}$. Equivalently, any invertible spectrum is equivalent to a suspension of the sphere spectrum $\mathbb{S}$.

We are most interested in Picard groups in homotopy theory, so let $(\mathcal{C}, \otimes, \mathbb{1})$ be a symmetric monoidal $\infty$-category. The **Picard space** $\mathcal{P}ic(\mathcal{C})$ is the $\infty$-groupoid of invertible objects in $\mathcal{C}$ and equivalences between them. For example, if $R$ is a commutative ring spectrum (in the sense of brave new algebra), then $(\mathrm{Mod}(R), \otimes, R)$ is a stable symmetric monoidal $\infty$-category. The homotopy groups of $\mathcal{P}ic(R)$ are well-understood:

$$\pi_*(\mathcal{P}ic(R)) \cong \begin{cases} \mathrm{Pic}(R) & * = 0 \\ (\pi_0(R))^\times & * = 1 \\ \pi_{*-1}(GL_1(R)) \cong \pi_{*-1}(R) & * \geq 2 \end{cases}$$

The Picard space generalizes the construction of the Picard group, and it is easier to compute, as it commutes with limits and filtered colimits [MS16]. In particular, it is compatible with descent methods.

## Galois Descent and the Stable Module Category

For a finite group $G$ and a field $k$ of characteristic $p > 0$, where $p$ divides the order of $G$, one can form the stable module category $\mathrm{StMod}(kG)$ as the localization of the category of $kG$-modules with respect to the module homomorphisms that factor through projective $kG$-modules. This is a stable symmetric-monoidal $\infty$-category, and so $\mathrm{StMod}(kG)$ can be characterized as the category of modules over a “ring spectrum with several objects" [SS03]. If $\mathrm{StMod}(kG)$ is compactly generated by the trivial module (for example, if $G$ is a $p$-group), then there is an equivalence of symmetric monoidal $\infty$-categories [SS03] [Mat15b]

$$\mathrm{StMod}(kG) \cong \mathrm{Mod}(k^{tG})$$

where $k^{tG}$ is a commutative ring spectrum called the Tate construction of $G$. Observe that $k^{tG}$ is related to group cohomology $H^*(G;k)$ through a variant known as **Tate cohomology**: $\widehat{H}^*(G;k) \cong \pi_{-*}(k^{tG})$.

This gives insight to how one might use Galois descent [MS16] [GL16]. In stable homotopy theory, the program of *brave new algebra* seeks to generalize notions in ring theory to ring *spectra*. In particular, a map $f : R \to S$ of commutative ring spectra is called a $G$-**Galois extension** [Rog08] if there is a action of $G$ on $S$ such that the natural maps $i : R \to S^{hG}$ and $h : S \otimes_R S \to F(G_+, S)$ are weak equivalences. A Galois extension is **faithful** if $S$ is faithful as an $R$-module. That is, if $M$ is an $R$-module such that $S \otimes_R M$ is contractible, then $M$ is contractible. Given a faithful Galois extension $f : R \to S$, we have Galois descent for $\mathrm{Mod}(R)$. That is, there is an equivalence of $\infty$-categories $\mathrm{Mod}(R) \cong \mathrm{Mod}(S)^{hG}$ [MS16]. As a corollary, there is an equivalence of spaces

$$\mathcal{P}ic(R) \cong \tau_{\geq 0}(\mathcal{P}ic(S)^{hG})$$

Therefore, in the setting of Galois descent, one can use the HFPSS to compute Picard groups.

# Current Research

## Quaternion Groups

One might hope that for a normal subgroup $K \trianglelefteq G$, the induced map of ring spectra $k^{tG} \to k^{tK}$ is a faithful $G/K$-Galois extension. In work with van de Meer, I show that this holds for $G = Q_{2^n}$ (generalized) quaternion, with normal subgroup the center $H \cong \mathbb{Z}/2$. This allows us to use Galois descent to compute $\mathrm{Pic}(\mathrm{StMod}(kQ))$. We also use these techniques for $G = C_{p^n}$ a cyclic $p$-group.

In general, to show that a $G$-Galois extension $R \to S$ is faithful, it is equivalent to show that the Tate construction $S^{tG}$ is contractible [Rog08]. Therefore, one can use the **Tate spectral sequence** to compute the homotopy groups of the Tate construction $S^{tG}$ and deduce whether or not a particular Galois extension is faithful. In the case of quaternion groups, we have:

**Proposition 1.1** (van de Meer-W. [vdMW21])**.** *Let $Q$ be a quaternion group, with center $H \cong \mathbb{Z}/2$. Then $k^{tQ} \to k^{tH}$ is a faithful $Q/H$-Galois extension.*

As a consequence, we obtain the homotopy fixed point spectral sequence:

$$H^s(Q/H; \pi_t(\mathcal{P}ic(k^{tH}))) \Rightarrow \pi_{t-s}(\mathcal{P}ic(k^{tH})^{hQ/H})$$

which converges to $\mathrm{Pic}(\mathrm{StMod}(kQ))$ for $t - s = 0$. Since we understand $\pi_*(\mathcal{P}ic(k^{t\mathbb{Z}/2}))$, we know most of the relevant differentials in this spectral sequence. The main difficulty is in computing the differentials originating from the $t - s = 0$ column. An analysis of these differentials shows that the kernel depends precisely on the existence of the cube root of unity $\omega$ in the $G \cong Q_8$ case.

**Theorem 1.2** (van de Meer-W. [vdMW21], cf Carlson-Thévenaz [CT00])**.** *Let $p = 2$, and let $\omega$ denote a cube root of unity.*

$$\mathrm{Pic}(\mathrm{StMod}(kQ_{2^n})) \cong \begin{cases} C_4 & \text{if } n = 3 \text{ and } \omega \notin k \\ C_4 \oplus C_2 & \text{if } n = 3 \text{ and } \omega \in k \\ C_4 \oplus C_2 & \text{if } n \geq 4 \end{cases}$$

## Groups with periodic cohomology

Recall that a group $G$ is said to have **periodic cohomology** (with period $n$) if there is an isomorphism of group cohomology $H^i(G; \mathbb{Z}) \cong H^{i+n}(G; \mathbb{Z})$ for all $i > 0$, where $\mathbb{Z}$ has the trivial $G$-action. A group $G$ has periodic cohomology if and only if all the abelian subgroups of $G$ are cyclic. These groups have been classified completely by work of Suzuki-Zassenhaus. Note that $C_{p^n}$ and $Q_{2^n}$ are precisely the $p$-groups with periodic cohomology.

If $G$ is not s $p$-group, there is not necessarily an equivalence $\mathrm{StMod}(kG) \cong \mathrm{Mod}(k^{tG})$. That is, $\mathrm{StMod}(kG)$ is not always compactly generated by the trivial module. There are two issues to consider: Observe that we can decompose the group algebra $kG$ into indecomposable two-sided ideals called **blocks**.

$$kG = A_1 \oplus \cdots \oplus A_r$$

which correspond to a decomposition of the identity $1 = e_1 + \cdots + e_r$ into a sum of primitive orthogonal central idempotent elements. This allows us to decompose the category of $\mathrm{Mod}(kG)$ into blocks. The block to which the trivial module $k$ belongs is called the **principal block**.

To pass from $\mathrm{Mod}(kG)$ to $\mathrm{StMod}(kG)$, we quotient out the projective modules. Observe that in order for $\mathrm{StMod}(kG)$ to be compactly generated by the trivial module, all non-principal blocks must be projective (equivalently, simple). Furthermore, we must also have that the principal block must be compactly generated by the trivial module. Work of Benson [Ben95] gives a condition on $G$ for when the latter condition holds, but it is not known for the class of groups $G$ when the former condition holds.

However, in my current work, I have shown that for these groups with periodic cohomology, both conditions hold, and so we have an equivalence $\mathrm{StMod}(kG) \cong \mathrm{Mod}(k^{tG})$. Moreover, given a normal cyclic subgroup $C_p$ of $G$, I showed that we have faithful Galois extensions $k^{tG} \to k^{tC_p}$. Thus, we can again compute $\mathrm{Pic}(\mathrm{StMod}(kG))$ using the HFPSS for groups with periodic cohomology.

**Theorem 1.3** (W., cf Carlson-Mazza-Nakano)**.** *Let $S_3$ be the symmetric group on 3 elements.*

$$\mathrm{Pic}(\mathrm{StMod}(kS_3)) \cong \begin{cases} \{e\} & \text{if } char(k) = 2 \\ C_4 & \text{if } char(k) = 3 \end{cases}$$

## Directions for Future Research

I propose three directions for future research. The first is to extend these computations to different classes of groups, and to study the more general Dade group of endo-permutation modules. The second is to study Picard groups for interesting variants of $\mathrm{StMod}(kG)$. In particular, the work of Benson-Iyengar-Krause [BIK08a] [BIK08b] computed the thick and localizing tensor-ideal subcategories of $\mathrm{StMod}(kG)$. In analogy to the computations of Picard groups in chromatic homotopy theory, one expects that the Picard groups of these subcategories to have deep representation-theoretic content. Finally, the third direction is to apply my computational techniques to other areas in (equivariant) stable homotopy theory.

### Generalizations

I propose two major ways to generalize these computations:

1. I propose to extend my computations to other classes of groups using descent with respect to families of elementary abelian subgroups of $G$ [MNN15]. In general, one has a descendable object $\mathcal{A} \in \mathrm{Mod}(k^{tG})$ [CT00] [Mat16] with the associated **descent spectral sequence**. Furthermore, we have a homotopy limit decomposition

$$\mathrm{StMod}(kG) \simeq \mathrm{holim}_{G/H \in \mathcal{O}_{\mathcal{A}}(G)^{op}} \mathrm{StMod}(kH)$$

   where $\mathcal{O}_{\mathcal{A}}(G)^{op}$ is determined by the family of elementary abelian subgroups of $G$. This is a generalization of Galois descent, where the descendable object is $k^{tH} \in \mathrm{Mod}(k^{tG})$, we have an equivalence $\mathrm{StMod}(kG) \simeq (\mathrm{StMod}(kH))^{hG/H}$, and the associated descent spectral sequence is exactly the HFPSS.

   I propose to compute $\mathrm{Pic}(\mathrm{StMod}(kG))$ for $G = D_{2^n}$ and $G = SD_{2^n}$, the dihedral and semi-dihedral groups. I have work in progress with van de Meer in understanding $\mathcal{O}_{\mathcal{A}}(G)^{op}$ for these families. Furthermore, the work of Carlson-Thévenaz [CT00] showed that for $G$ semi-dihedral, there are torsion elements in the Picard group. Just as in the quaternion case, I expect the homotopy-theoretic approach to provide insight into these torsion elements.

   I further propose to compute the Picard groups for extraspecial and almost-extraspecial $p$-groups. The subgroup structure of these groups is well understood, and so I expect good control over $\mathcal{O}_{\mathcal{A}}(G)^{op}$ and descent. This would be a major advancement in a purely homotopical approach to the computation of Picard groups for arbitrary $p$-groups, as the classical work of Carlson-Thévenaz [CT05] reduces to these special cases.

2. In modular representation theory, for $P$ a $p$-group, one is interested also in the **Dade group** $D(P)$, that is, the group of **endo-permutation modules** - the $kP$-modules $M$ where $\mathrm{End}_k(M) \cong k \oplus kY$, where $Y$ is a $P$-set [Dad78]. These endo-permutation modules are crucial in representation theory as sources of simple modules for $p$-solvable groups, and in understanding Morita or derived equivalences.

Note that endo-trivial modules are endo-permutation modules, so the study of $D(P)$ is closely related to the study of Pic(StMod($kP$)) [CT00].

I propose to categorify the Dade group, by constructing a Dade *space* and studying it via descent methods. The Dade group can be understood in terms of Bouc functors [BT00], which are closely related to Mackey functors in equivariant stable homotopy theory. The Dade group can also be defined in terms of Dade algebras, which are split central simple algebras with a $P$-invariant basis and at least one $P$-fixed point. Therefore, the study of Dade groups is also closely related to the study of Brauer groups.

Brauer groups have been similarly categorified into Brauer spaces, and have been studied via descent methods [BRS12] [AG14] [GL16]. Computing the Brauer group of StMod($kP$) is also an interesting open problem, and my prior computations of Pic(StMod($kG$)) should provide some insight into the Brauer group of StMod($kG$).

## Picard Groups of Thick Subcategories

The ideas of nilpotence and descent are intimately related to the study of stable symmetric monoidal $\infty$-categories. For example, the Nilpotence theorem [DHS88] classified the thick tensor-ideal subcategories of the category of compact spectra $\mathrm{Sp}^{fin}$. These subcategories are precisely detected by the Morava K-theories $K(n,p)$, and are thus intimately related to chromatic homotopy theory.

For an arbitrary stable symmetric monoidal $\infty$-category $\mathcal{C}$, one studies **tensor-triangular geometry** [Bal10]. That is, the classification of thick tensor-ideal subcategories $\mathcal{X}$ of the compact objects in $\mathcal{C}$. $\mathcal{X}$ is said to be **tensor-ideal** if $X \otimes Y \in \mathcal{X}$ implies either $X \in \mathcal{X}$ or $Y \in \mathcal{X}$. A full subcategory $\mathcal{X} \subset \mathcal{C}$ is **thick** if $\mathcal{X}$ is closed under finite limits and colimits, and also closed under retracts. One can also study the localizing tensor-ideal subcategories of $\mathcal{C}$, where a **localizing** subcategory is a thick subcategory that is closed under arbitrary colimits.

Much like in algebraic geometry, these thick tensor-ideal subcategories can be assembled with an interesting topology, known as the **Balmer spectrum**. Much work has been done for the derived category [NB92], modules over ring spectra [Mat15a] and equivariant stable homotopy theory [BS17], [BHN$^+$19]. In particular, work of Benson-Iyengar-Krause [BIK08b] classified the thick (tensor-ideal) subcategories for the stable module category StMod($kG$), which is determined by group cohomology $H^*(G;k)$. Specifically, they correspond to specialization-closed subsets of the variety Proj($H^*(G;k)$).

I propose to study these thick (tensor-ideal) subcategories of StMod($kG$) by computing their Picard groups. The simplest non-trivial case is where $k = \mathbb{F}_2$ and $G = \mathbb{Z}/2 \times \mathbb{Z}/2$. Here, the thick (tensor-ideal) subcategories correspond to homogenous prime ideals of $H^*((\mathbb{Z}/2)^2;\mathbb{F}_2) \cong \mathbb{F}_2[x_1,x_2]$. The associated (co)localization functors have nice properties, and are given explicitly by work of Benson-Iyengar-Krause [BIK08b]. I currently have work in progress understanding these functors in terms of Mod($k^{t(\mathbb{Z}/2)^2}$), using the local cohomology spectral sequence [BHV18]. I expect that using descent methods will prove effective in computing these local Picard groups.

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