FALL 2022 32AH, CHALLENGE PROBLEM REPORT 1

Abstract. In this Challenge Problem Report, you will explore the geometry of linear maps $\mathbb{R}^2 \mapsto \mathbb{R}^2$. To complete the first challenge problem report, you will write up solutions to the assigned problems. Your write-up should include exposition in your own words, and read like a chapter or section of a textbook. Be sure to clearly label your answers to the questions.

1. Linear maps and basis vectors

In lecture, we have abstractly defined maps of vector spaces, and characterized linear maps of the form $\mathbb{R} \to \mathbb{R}$. In this challenge problem report, you will explore the geometry of linear maps $\mathbb{R}^2 \mapsto \mathbb{R}^2$. The basis of this exploration is the following theorem:

Theorem 1.1. *Let V be a vector space with basis $\{\boldsymbol{v_1}, \cdots, \boldsymbol{v_n}\}$, and let W be an arbitrary vector space. A linear map $T : V \to W$ is determined by what it does on basis vectors.*

For example, let us consider the vector space $\mathbb{R}^2$, with the standard basis. Then we can write the vector $\boldsymbol{v} = \langle x, y\rangle \in \mathbb{R}^2$ in terms of the standard basis:

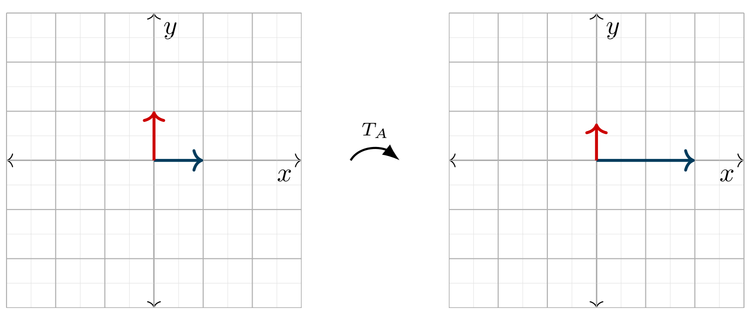
$$\boldsymbol{v} = x\boldsymbol{e_1} + y\boldsymbol{e_2}$$

Let us consider a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$. Observe that using linearity, we have that

$$\begin{aligned} T(\boldsymbol{v}) &= T(x\boldsymbol{e_1} + y\boldsymbol{e_2}) \\ &= xT(\boldsymbol{e_1}) + yT(\boldsymbol{e_2}) \end{aligned}$$

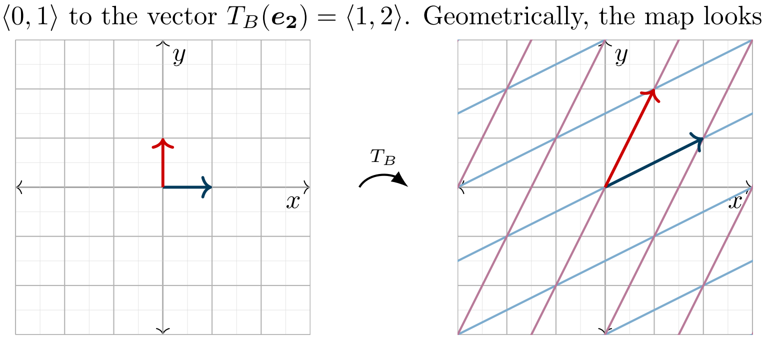
Thus, if we know the values of $T(\boldsymbol{e_1})$ and $T(\boldsymbol{e_2})$ (e.g. how T acts on the basis vectors), then we can use the above formula to calculate $T(\boldsymbol{v})$ for any vector $\boldsymbol{v} \in \mathbb{R}^2$.

Consider the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & \frac{3}{4} \end{bmatrix}$. From what we saw in class, the map $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_A(\boldsymbol{v}) = A\boldsymbol{v}$ is linear. We can compute that the linear map T_A sends the vector $\boldsymbol{e_1} = \langle 1, 0\rangle$ to the vector $T_A(\boldsymbol{e_1}) = \langle 2, 0\rangle$, and the vector $\boldsymbol{e_2} = \langle 0, 1\rangle$ to the vector $T_A(\boldsymbol{e_2}) = \langle 0, \frac{3}{4}\rangle$. Geometrically, the map looks like this:



It follows from theorem 1.1 that $T_A(\langle x, y\rangle) = \langle \frac{x}{2}, 2y\rangle$. In other words, the linear map T_A stretches the x-component by a factor of 2, and shrinks the y-component by a factor of $\frac{3}{4}$.

Not every linear map is as simple, however! Let us consider the linear map associated to the matrix $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. We can compute that the linear map T_B sends the vector $\boldsymbol{e_1} = \langle 1, 0\rangle$ to the vector $T_B(\boldsymbol{e_1}) = \langle 2, 1\rangle$, and the vector $\boldsymbol{e_2} = \langle 0, 1\rangle$ to the vector $T_B(\boldsymbol{e_2}) = \langle 1, 2\rangle$. Geometrically, the map looks like this:



We observe that the linear map T_B sends the unit square spanned by $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$ to the parallelogram spanned by $T_B(\boldsymbol{e_1})$ and $T_B(\boldsymbol{e_2})$. By linearity, T_B then sends the grid lines to a grid of congruent parallelograms.

In general, T_B is a good mental image for the geometric interpretation of a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$. This is why linear maps are sometimes called linear transformations - as they transform the vectors in the domain into (potentially different) vectors in the range.

In this first section, you will explore other examples of linear maps $T : \mathbb{R}^2 \to \mathbb{R}^2$.

(1) Find a 2×2 matrix R such that the associated linear map $T_R : \mathbb{R}^2 \to \mathbb{R}^2$ reflects vectors across the x-axis.

(2) Fix an angle θ. Sketch and describe the linear map T_θ associated to the matrix

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

2. Eigenvectors and Eigenvalues

Given a matrix $M \in M_{2\times 2}(\mathbb{R})$, what other information can we extract about its associated linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$?

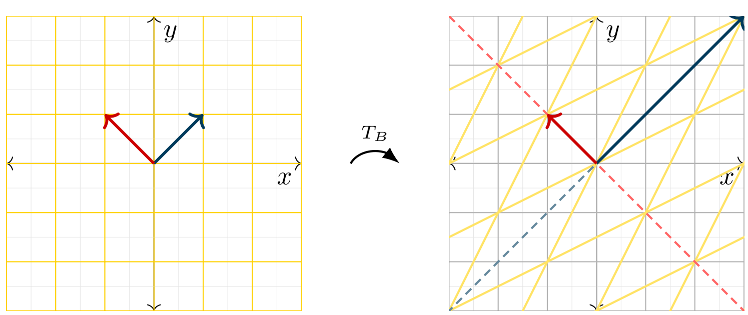
If we again consider the linear map corresponding to the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & \frac{3}{4} \end{bmatrix}$, we observed that $T_A(\boldsymbol{e_1}) = \langle 2, 0\rangle = 2\boldsymbol{e_1}$, and that $T_A(\boldsymbol{e_2}) = \langle 0, \frac{3}{4}\rangle = \frac{3}{4}\boldsymbol{e_2}$. That is, $T(\boldsymbol{e_i})$ was parallel to $\boldsymbol{e_i}$. Furthermore, these scalars correspond precisely to how T stretched and shrunk the x and y components.

On the other hand, the linear map T_B corresponding to the matrix $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ changes the direction of the standard basis vector, so $T(\boldsymbol{e_i})$ is **not** parallel to $\boldsymbol{e_i}$. However, one might still ask the following question:

Is there a vector $\boldsymbol{v} \in \mathbb{R}^2$ such that $T(\boldsymbol{v})$ is parallel to $\boldsymbol{v}$?

Definition 2.1. Let T be the linear map associated to a matrix M. (That is, $T(\boldsymbol{v}) = M\boldsymbol{v}$).
We say that a non-zero vector $\boldsymbol{v}$ is an **eigenvector of** M, with **eigenvalue** λ if $M\boldsymbol{v} = \lambda\boldsymbol{v}$.

For example, if we consider the linear map T_B corresponding to the matrix $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, we observe by inspection that if we set $\boldsymbol{v} = \langle 1, 1\rangle$, we have that $T_B(\boldsymbol{v}) = 3\boldsymbol{v}$. Similarly, if we set $\boldsymbol{w} = \langle -1, 1\rangle$, we have that $T_B(\boldsymbol{w}) = \boldsymbol{w}$. Thus, $\boldsymbol{v}$ and $\boldsymbol{w}$ are eigenvectors of T_B. In fact, any scalar multiple of $\boldsymbol{v}$ or $\boldsymbol{w}$ will be eigenvectors as well.



Observe that in this case, the eigenvectors $\{\langle 1, 1\rangle, \langle -1, 1\rangle\}$ form a basis for $\mathbb{R}^2$! (You will see in the exercises that in general, the eigenvectors do not necessarily form a basis). Thus, we can write a vector

$\boldsymbol{v} = a\langle 1,1\rangle + b\langle -1,1\rangle$. Then as before,

$$\begin{aligned} T(\boldsymbol{v}) &= T(a\langle 1,1\rangle + b\langle -1,1\rangle) \\ &= aT(\langle 1,1\rangle) + bT(\langle -1,1\rangle) \\ &= 3a\langle 1,1\rangle + b\langle -1,1\rangle \end{aligned}$$

This allows us to describe the geometric properties of the linear map T_B: it stretches vectors by a factor of 3 in the $\langle 1,1\rangle$ direction, and does nothing to the part in the $\langle -1,1\rangle$ direction.

To summarize, the eigenvectors associated to a matrix M are the vectors whose lines are preserved by the linear transformation T_M. Their corresponding eigenvalues describe how the eigenvectors are stretched by the linear transformation T_M.

Given a matrix, how can one algebraically find its eigenvectors?

To answer this question properly, we will need to learn a bit more linear algebra machinery (see the bonus food for thought section, and the course lecture notes). For now, we will use the following tool:

Definition 2.2. Let $M \in M_{2\times 2}(\mathbb{R})$ be the matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The **characteristic polynomial** of M is a polynomial $p(\lambda)$, given by the expression

$$p(\lambda) = (a-\lambda)(d-\lambda) - bc$$

Theorem 2.3. *The roots of the characteristic polynomial are precisely the eigenvalues of M.*

Once we find the eigenvalues of M, we can find the eigenvectors associated to the eigenvalue λ_i by solving the system of linear equations $M\boldsymbol{v} = \lambda_i \boldsymbol{v}$.

For example, the characteristic polynomial of $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ is the polynomial $(2-\lambda)(2-\lambda) - 1 * 1 = \lambda^2 - 4\lambda + 3$. The roots of this polynomial are precisely $\lambda_1 = 3$ and $\lambda_2 = 1$. We can then solve the system of linear equations

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ 3y \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

to determine that the associated eigenvectors are parallel to $\langle 1,1\rangle$, and $\langle -1,1\rangle$, respectively.

Remark 2.4. Observe that the eigenvalues of a 2×2 matrix need not be distinct, and they may be complex numbers. In the exercises, you will figure out what these situations correspond to geometrically.

In this section, you will use eigenvectors and eigenvalues to study linear maps.

(3) Sketch and describe the linear map T_S associated to the matrix

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

.

What are the eigenvectors of S?

(4) For what values of θ does the matrix

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

have real eigenvalues? Prove your answer.

(5) Fix an angle θ. Does the linear map $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ associated to the matrix

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

have any eigenvectors? (**Hint:** Your answer may change based on certain values of θ.)

(6) Let $a, b \in \mathbb{R}$ such that a and b are not both zero. Sketch and describe the linear map T_M associated to the matrix

$$M = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

What are the eigenvectors of S?

3. Bonus Food for Thought

These questions/reading are optional, and do not need to be answered for full credit. You should revisit them once we learn about determinants.

First, let us see where the formula for the characteristic polynomial $p(\lambda) = (a-\lambda)(d-\lambda) - bc$ comes from: Observe that we can write the vector equation $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \boldsymbol{v} = \lambda \boldsymbol{v}$ as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \boldsymbol{v} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{v}$$

By properties of matrices, this is equivalent to studying the equation of the form

$$\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} \boldsymbol{v} = \mathbf{0}$$

Since by definition, an eigenvector cannot be the $\mathbf{0}$ vector, this tell us that the linear map T_λ associated to the matrix $\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$ sends a non-zero vector to a zero vector. In other words, T_λ cannot be an invertible linear map (because it is not injective). When we study determinants, we will learn that a linear map is not invertible if and only if its standard matrix has determinant equal to 0. We see immediately that the determinant of $\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$ is precisely our definition of the characteristic polynomial.

(A) How might you generalize these ideas to study the eigenvalues and eigenvectors of linear maps $\mathbb{R}^3 \to \mathbb{R}^3$? $\mathbb{R}^n \to \mathbb{R}^n$?

(B) Can you use eigenvalues and eigenvectors to classify the types of linear maps $\mathbb{R}^2 \to \mathbb{R}^2$?
  (a) There are 3 ways to classify the eigenvalues of a 2×2 matrix M:
    - The eigenvalues are real and equal.
    - The eigenvalues are real, and unequal.
    - The eigenvalues are complex conjugates (e.g. of the form $a \pm bi$).
  (b) For each type above, we can then further subdivide by the number of eigendirections (e.g. the number of non-parallel eigenvectors):
    - No eigenvectors.
    - One eigendirection.
    - Two eigendirections.
    - All vectors are eigenvectors.

  Find matrices that correspond to these subdivisions, and sketch and describe their linear transformations.

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