# 32B CHALLENGE PROBLEM REPORT 2

SPRING 2023

Abstract. In this Challenge Problem set, you will study the Möbius strip as an example of a non-orientable surface.

## 1. The Möbius strip

We saw in class that in order to integrate a scalar function over some surface $S$, we only needed a **parametrization** of $S$. However, in order to integrate a vector field over some surface $S$, we need more information - we need an **orientation** of $S$.

However, as we discussed in class, it is not always possible to possible to orient surfaces!

**Definition 1.1.** A surface $S$ that cannot be oriented is called a **non-orientable surface**. That is, it is not possible to choose a unit normal vector on $S$ in a continuously varying manner.

Intuitively, this means that there exists a closed curve $C$ on $S$ where it is not possible to consistently choose $\boldsymbol{n}$ on $C$.

In other words, if you start at a point $P$ on the closed curve $C$ with chosen normal vector $\boldsymbol{n}(P)$, then if you transport $\boldsymbol{n}(P)$ while moving along $C$, after you have completed one loop around $C$, then the direction of the transported vector $\boldsymbol{n}(P)$ will point in the direction $-\boldsymbol{n}(P)$.

**Example 1.2.** The classic example of a non-orientable surface is the Möbius strip:

You can construct a Möbius strip with a rectangular strip of paper by joining the two ends together with a 180 degree twist. This creates a one-sided surface, and you will show in the exercises why this surface is not orientable.

*Remark* 1.3. In fact, a surface is non-orientable if and only if it contains a Möbius strip.

As a consequence, it is not possible to compute surface integrals of vector fields $\boldsymbol{F}$ on the Möbius strip, since flux is meaningless without orientation (in other words, we need an orientation to make sense of $\boldsymbol{F} \cdot \boldsymbol{n}$).

In fact, you will show why the flux integral is not well-defined in the exercises below.

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To complete the second challenge problem set, you will write up solutions to the following problems. Your write-up should include exposition and look like a chapter or section of a textbook. Be sure to clearly label your answers to the questions.

(1) Sketch the Möbius strip (denoted $M$) parametrized by

$$G(u,v) = \left\langle \left(1 + v\cos(\frac{u}{2})\right)\cos(u), \left(1 + v\cos(\frac{u}{2})\right)\sin(u), v\sin(\frac{u}{2}) \right\rangle$$

on the domain $D = \{(u,v) \mid 0 \le u \le 2\pi, -\frac{1}{2} \le v \le \frac{1}{2}\}$.

You can do this by hand, or include a picture using graphing software.

  (a) Let $C$ denote the intersection of $M$ with the $xy$-plane. What is $C$?

(2) Compute the normal vector for $v = 0$. That is, compute $\boldsymbol{N}(u,0)$.

  (a) Sketch $\boldsymbol{N}(u,0)$ on $M$.
  (b) Observe that $\boldsymbol{N}(u,0)$ varies continuously for $0 \le u \le 2\pi$, moving once along $C$. However, show that $\boldsymbol{N}(2\pi,0) = -\boldsymbol{N}(0,0)$.
  (c) Explain why this implies that $M$ is not orientable.

(3) Consider the vector field $\boldsymbol{F} = \langle 0,0,1\rangle$.

  (a) Use the paramterization $G(u,v)$ on the domain $D = \{(u,v) \mid 0 \le u \le 2\pi, -\frac{1}{2} \le v \le \frac{1}{2}\}$ to compute the vector surface integral of $\boldsymbol{F}$ across the Mobius strip $M$.
  (b) Use the paramterization $G(u,v)$ on the domain $D = \{(u,v) \mid \frac{\pi}{2} \le u \le \frac{5\pi}{2}, -\frac{1}{2} \le v \le \frac{1}{2}\}$ to compute the vector surface integral of $\boldsymbol{F}$ across the Mobius strip $M$.
  (c) Should your answers be the same, or different?

## 2. Bonus Food for Thought

**This section has optional content, and the questions in this section do not need to be answered for full credit.**

From the previous exercises, we saw that one can parametrize the Möbius strip, and thus it is possible to calculate surface integrals of scalar functions on the Möbius strip (since we can make sense of $f(G(u,v)||\boldsymbol{N}(u,v)||)$. For example, you will compute the surface area of a specific Möbius strip in the exercises below.

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Consider the Möbius strip (denoted $M$) parametrized by

$$G(u,v) = \left\langle \left(1 + v\cos(\frac{u}{2})\right)\cos(u), \left(1 + v\cos(\frac{u}{2})\right)\sin(u), v\sin(\frac{u}{2})\right\rangle$$

on the domain $D = \{(u,v) \mid 0 \le u \le 2\pi, -\frac{1}{2} \le v \le \frac{1}{2}\}$.

(A) Write an integral that computes the surface area of Möbius strip $M$, and evaluate it to four decimal places.

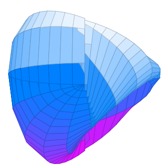
(B) Suppose that the density of the Möbius strip $M$ is $x^2 + y^2 + z^2$. Calculate the mass of the Mobius strip (evaluating it to four decimal places).

In this section, we study some examples of other non-orientable surfaces.

**Example 2.1.** The **Roman surface** is a non-orientable surface parametrized by

$$G(\theta,\phi) = \langle\cos(\phi)\sin(\phi)\sin(\theta), \cos(\phi)\sin(\phi)\cos(\theta), \cos^2(\phi)\cos(\theta)\sin(\theta)\rangle$$

on the domain $D = \{(\theta,\phi) \mid 0 \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{2}\}$.



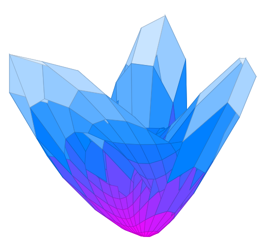
It is named the Roman surface because it was discovered by Jakob Steiner when he was in Rome in 1844. Points on the Roman surface satisfy the equation

$$x^2y^2 + y^2z^2 + z^2x^2 - 4xyz = 0$$

**Example 2.2.** Boy's surface is a non-orientable surface parametrized by

$$G(\theta,\phi) = \left\langle \frac{\sqrt{2}\cos^2(\phi)\cos(2\theta) + \cos(\theta)\sin(2\phi)}{2 - \sqrt{2}(\sin(3\theta)\sin(2\phi)}, \frac{\sqrt{2}\cos^2(\phi)\sin(2\theta) - \sin(\theta)\sin(2\phi)}{2 - \sqrt{2}(\sin(3\theta)\sin(2\phi))}, \frac{3\cos^2(\phi)}{2 - \sqrt{2}(\sin(3\theta)\sin(2\phi))}\right\rangle$$

on the domain $D = \{(\theta,\phi) \mid \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}, 0 \le \phi \le \pi\}$.



It is named Boy's surface because it was discovered by Werner Boy in 1901. Boy was able to find several models of this surface, but a parametrization was not found until much later (by Apéry in 1986)

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