Applications of Group Cohomology to 3-Manifolds

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Slides can be found at
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Let $G$ be a topological group.

Is it possible to construct a space $X$ such that $\pi_1(X) \cong G$?

If $G$ is discrete, then yes! We can use covering space theory.

We need a simply connected space $Y$ such that $G$ acts freely on $Y$. Then $Y \xrightarrow{p} X = Y/G$ is a universal covering space, and we have

$$G \cong \pi_1(X)/p_*(\pi_1(Y)) \cong \pi_1(X)$$

We can do even better: For discrete $G$, we can build a space $K(G, 1)$ such that $\pi_n(K(G, 1)) \cong \begin{cases} G & n = 1 \\ 0 & \text{else} \end{cases}$

We just need a contractible space $Y$ such that $G$ acts freely on $Y$. 

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How does this generalize to arbitrary topological groups? One can construct $BG$, the *classifying space* of $G$. Unfortunately, this is no longer a $K(G, 1)$.

**Construction**

*One construction is the Milnor construction, which constructs $EG$, a contractible CW complex such that $G$ acts freely.*

$$EG = \text{colim}_i G^i$$

Then $BG = EG/G$, the quotient space of the $G$-action.

*One can think of this as generalization of covering space theory.*
### Example

\[ E\mathbb{Z} = \mathbb{R}, \quad B\mathbb{Z} \cong S^1 \]

### Example

\[ E\mathbb{Z}/2 = S^\infty, \quad B\mathbb{Z}/2 \cong \mathbb{R}P^\infty \]

### Example

If \( G \) is a discrete group, then \( BG \cong K(G, 1) \).

### Example

\[ BS^1 \cong \mathbb{C}P^\infty \]

### Example

\[ B(G \times H) \cong BG \times BH \]
BG is a nice object to study. It is called the classifying space since

\[ [X, BG] = \{ \text{Isomorphism classes of principal } G\text{-bundles over } X \} \]

**Definition**

Recall that a principal \( G \)-bundle over \( X \) is a fiber bundle \( \pi : P \to X \) with fiber \( G \), where \( G \) acts on itself by (left) translations.

In particular, \( G \) acts freely on \( P \), and we always have a fibration \( P \to X \to BG \).

**Example**

\( EG \to BG \) is the universal principal bundle.
Definition

The group cohomology of \( G \) is defined to be the cohomology of \( BG \):

\[
H^\ast(G; \mathbb{Z}) := H^\ast(BG; \mathbb{Z})
\]

More generally, given a \( \mathbb{Z} \)-module \( M \), one can define

Definition

The group cohomology of \( G \) with coefficients in \( M \) is defined to be the cohomology of \( BG \):

\[
H^\ast(G; M) := H^\ast(BG; M)
\]
### Example

\[ H^i(\mathbb{Z}/2; \mathbb{Z}) := H^i(\mathbb{R}P^\infty; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i \text{ odd} \geq 1 \\ \mathbb{Z}/2 & i \text{ even} \geq 2 \end{cases} \]

### Example

\[ H^i(\mathbb{Z}/2; \mathbb{F}_2) := H^i(\mathbb{R}P^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[x], \text{ with } |x| = 1 \]
What if $G$ acts (linearly) on $M$ in an interesting way?

**One answer:** Group cohomology with local coefficients. This takes into account the action of $\pi_1(BG) = G$ on $M$.

A $\mathbb{Z}$-module with $G$-action is the same as a $\mathbb{Z}G$-module. So when considering the category of modules with $G$ action, one can instead consider the category of $\mathbb{Z}G$-modules.

For example, one defines the cohomology of $BG$ with local coefficients $M$ to be

$$H^*(G; M) := H^*(\text{Hom}_{\mathbb{Z}G}(C_n(EG), M))$$
What if $G$ acts (linearly) on $M$ in an interesting way?

**Another answer:** Homological algebra:

Since $EG$ is contractible, and $G$ acts freely on $EG$, then we have that $C_*(EG)$ is a **free resolution** of $\mathbb{Z}$ over $\mathbb{Z}G$.

Therefore, we have that

$$H^*(G; M) \cong \text{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, M)$$

This gives us a purely algebraic way of understanding group cohomology.
\( G = \mathbb{Z}/n, \mathbb{Z} \) and \( \mathbb{F}_p \) with trivial action.

**Example**

\[
\cdots \xrightarrow{\cdot (\Sigma g^i)} \mathbb{Z} G \xrightarrow{\cdot (g-1)} \mathbb{Z} G \xrightarrow{\cdot (\Sigma g^i)} \mathbb{Z} G \xrightarrow{\cdot (g-1)} \mathbb{Z} G \xrightarrow{\epsilon} \mathbb{Z}
\]

\( H^i(\mathbb{Z}/n; \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z} & i = 0 \\
0 & i \text{ odd } \geq 1 \\
\mathbb{Z}/n & i \text{ even } \geq 2
\end{cases} \)

**Example**

\[
\cdots \xrightarrow{\cdot (\Sigma g^i)} \mathbb{F}_p G \xrightarrow{\cdot (g-1)} \mathbb{F}_p G \xrightarrow{\cdot (\Sigma g^i)} \mathbb{F}_p G \xrightarrow{\cdot (g-1)} \mathbb{F}_p G \xrightarrow{\epsilon} \mathbb{F}_p
\]

\( H^*(\mathbb{Z}/p; \mathbb{F}_p) \cong \mathbb{F}_p[x] \otimes \Lambda(y), \text{ with } |x| = 2 \text{ and } |y| = 1 \)

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Theorem (Künneth Formula)

Let $X$ and $Y$ be topological spaces and $F$ be a field. Then for each integer $k$ we have a natural isomorphism

$$\bigoplus_{i+j=k} H^i(X; F) \otimes H^j(Y; F)) \rightarrow H^k(X \times Y; F)$$

Example

If $k$ is a field of characteristic $p$, and $G = (\mathbb{Z}/p)^n$, then

$$H^*(G, k) = \begin{cases} \mathbb{F}_p[x_1, \ldots, x_n] & |x_i| = 1, \; p = 2 \\ \mathbb{F}_p[x_1, \ldots, x_n] \otimes \Lambda(y_1, \ldots, y_n) & |x_i| = 2, |y_i| = 1, \; p \neq 2 \end{cases}$$
Definition

Given a fibration $F \to X \to B$, we have the Serre spectral sequence:

$$E_2^{p,q} = H^q(B; H^p(F)) \Rightarrow H^{p+q}(X)$$

This spectral sequence is a computational tool whose inputs are $H^*(B)$ and $H^*(F)$. If we can also figure out some additional information (the differentials), then we can compute $H^*(X)$. 
### Example

A SES of groups $1 \to N \to G \to G/N \to 1$ yields a fibration $BN \to BG \to B(G/N)$. Given a $G$-module $M$, the associated spectral sequence is the Lyndon-Hochschild-Serre spectral sequence:

$$E_2^{p,q} = H^q(G/N; H^p(N; M)) \Rightarrow H^{p+q}(G; M)$$
From now on, let $G$ be a finite group. 

When does $G$ act freely on $S^n$?
Proposition

If \( n \) is even, then the only non-trivial finite group that can act freely on \( S^n \) is \( \mathbb{Z}/2 \).

Proof.

We have a group homomorphism \( \deg : G \to \mathbb{Z}/2 \) by taking the degree of the map \( S^n \xrightarrow{\cdot g} S^n \).

Since \( G \) acts freely, \( \cdot g \) is a fixed point free map for nontrivial \( g \). Therefore, by the hairy ball theorem, \( \cdot g \) is homotopic to the antipodal map.

Hence for nontrivial \( g \in G \), we have \( \deg(g) = -1 \). Hence \( \deg : G \to \mathbb{Z}/2 \) is injective. \qed
Warm up: Finite Groups acting freely on $S^1$:

**Proposition**

$\mathbb{Z}/n$ is the only finite group that acts freely on $S^1$.

**Proof.**

If $G$ is a finite group acting freely on $S^1$, then we have a fiber bundle

$$G \to S^1 \xrightarrow{p} S^1/G$$

However, we know that $S^1/G \cong S^1$. We then have by covering space theory that

$$G \cong \pi_1(S^1/G)/p_*(\pi_1(S^1)) \cong \mathbb{Z}/p_*(\mathbb{Z})$$
Let $n > 1$. If $G$ acts freely on $S^n$, then it again a covering space action, and so we again obtain a fiber bundle

$$G \to S^n \xrightarrow{p} S^n/G$$

$S^n/G$ is a closed manifold. Moreover, note that since $\pi_1(S^n) \cong 0$, we have by covering space theory

$$G \cong \pi_1(S^n/G)$$

How does group cohomology come into the picture?

**Definition**

A finite group $G$ is **periodic** of period $k > 0$ if

$$H^i(G; \mathbb{Z}) \cong H^{i+k}(G; \mathbb{Z})$$

for all $i \geq 1$, where $\mathbb{Z}$ has trivial $G$ action.
Proposition

If $G$ acts freely on $S^n$, then $G$ is periodic of period $n + 1$.

Proof.

If $n$ is even, this is true, since we saw $H^i(\mathbb{Z}/2; \mathbb{Z})$ is even periodic. So we need to prove the statement for $n$ odd. We consider the fibration

$$S^n \to S^n/G \to BG$$

Note that $\cdot g : S^n \to S^n$ is fixed point free, and hence has degree 1 (and is orientation preserving). So the action of $G$ on $H^*(S^n)$ is trivial. So we now compute the Serre spectral sequence:

$$E_2^{p,q} = H^q(BG; H^p(S^n; \mathbb{Z})) \Rightarrow H^{p+q}(S^n/G; \mathbb{Z})$$
The only non-trivial differential is a $d_3 : E^i_{2,3} \rightarrow E^{i+3+1,0}_2$.

Furthermore, $S^3/G$ is 3-dimensional, and hence $H^*(S^3/G) \cong 0$ for $* > 3$.

Therefore, $E^{p,q}_\infty = 0$ for $p + q > 3$. For example, the $H^5(G)$ in degree $(5,0)$ must be killed, and so $d_3 : H^1(G) \rightarrow H^5(G)$ must be surjective. But it must also be injective, since $H^1(G)$ in degree 4 cannot survive.
Proposition

G is periodic iff all the abelian subgroups of G are cyclic.

Theorem (Suzuki-Zassenhaus)

There are 6 families of periodic groups.

I $\mathbb{Z}/m \times \mathbb{Z}/n$ with $m, n$ coprime.

II $\mathbb{Z}/m \times (\mathbb{Z}/n \times Q_{2^k})$ with $m, n$, and 2 coprime.

III $((\mathbb{Z}/m \times \mathbb{Z}/n) \rtimes T_i)$ where $m, n$, and 6 coprime.

IV Groups coming from $TL_2(\mathbb{F}_3) \cong 2S_4$

V $(\mathbb{Z}/m \times \mathbb{Z}/n) \times SL_2(\mathbb{F}_p)$ with $m, n, (p^2 - 1)$ coprime, $p \geq 5$.

VI Groups coming from $TL_2(\mathbb{F}_p)$ for $p \geq 5$
Example

\(\mathbb{Z}/p \times \mathbb{Z}/p\) does not act freely on \(S^n\):

\[
H^i(\mathbb{Z}/p \times \mathbb{Z}/p; \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z} & i = 0 \\
(\mathbb{Z}/p)^{i-1} & i \text{ odd, } i \geq 1 \\
(\mathbb{Z}/p)^{i+2} & i \text{ even, } i \geq 2
\end{cases}
\]

One can deduce this using the Kunneth formula to calculate

\(H^*(\mathbb{Z}/p \times \mathbb{Z}/p; \mathbb{F}_p) \cong \mathbb{F}_p[x_1, x_2] \otimes \Lambda(y_1, y_2)\). We can then use the universal coefficient theorem to recover integral coefficients.
**Warning:** Not every periodic group with period 4 acts freely on $S^3$:

**Example**

$S_3$ has period 4, but does not act freely on $S^3$.

\[
H^i(S_3; \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z} & i = 0 \\
\mathbb{Z}/2 & i = 2 \mod 4 \\
\mathbb{Z}/6 & i = 0 \mod 4 \\
0 & i \text{ odd} 
\end{cases}
\]

Milnor showed that if a finite group $G$ acts freely on $S^n$, then every element of order 2 in $G$ is central.

**Theorem (Madsen-Thomas-Wall)**

A finite group $G$ acts freely on some sphere iff $G$ is periodic and every element of order 2 in $G$ is central.
Example

$\mathbb{Z}/p$ acts freely on $S^3$.
Consider the unit sphere $S^3 \subseteq \mathbb{C}^2$. Then for any $q$ coprime to $p$, $\mathbb{Z}/p$ acts by multiplication by

$$\begin{bmatrix}
e^{\frac{2\pi i}{p}} & 0 \\
0 & e^{\frac{2\pi qi}{p}}
\end{bmatrix}$$

The quotient manifolds $S^3/G$ are the 3-dimensional lens spaces $L(p; q)$. 
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III $(\mathbb{Z}/m \times \mathbb{Z}/n) \rtimes T_i$ where $m, n$, and 6 coprime.

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VI Groups coming from $TL_2(\mathbb{F}_p)$ for $p \geq 5$

Theorem (Madsen-Thomas-Wall)

A finite group $G$ acts freely on some sphere iff $G$ is periodic and every element of order 2 in $G$ is central.
Theorem (Wolf)

There are 5 families of finite groups that act freely on $S^3$:

- **Cyclic case**: $G \cong \mathbb{Z}/n$
- **Dihedral case**: $G \cong \langle x, y \mid xyx^{-1} = y^{-1}, x^{2m} = y^n \rangle$ for $m, n$ coprime, with $m \geq 1, n \geq 2$. For example, $Q_8$.
- **Tetrahedral case**: $G \cong \langle x, y, z \mid (xy)^2 = x^2 = y^2, zxz^{-1} = xy, z^{3k} = 1 \rangle$ for $m, 6$ coprime, with $k, m \geq 1$. For example, $2A_4$.
- **Octahedral case**: $G \cong 2S_4$
- **Icosahedral case**: $G \cong 2A_5$

And direct products of any of the above groups with a cyclic group of relatively prime order.
Example

In the cyclic case, the $S^3/G$ are Lens spaces.

Example

In the dihedral case, the $S^3/G$ are Prism manifolds.

Example

In the icosahedral case, $S^3/(2A_5)$ is the Poincare homology sphere.
All the 3-manifolds $S^3/G$ arising from those families have finite fundamental group:

$$\pi_1(S^3/G) \cong G$$

**Definition**

A 3-manifold is spherical if it is of the form

$$M = S^3/G$$

Classifying these was known as the spherical space form problem.

**Theorem (Elliptization conjecture)**

A 3-manifold with finite fundamental group is a spherical manifold.

This is equivalent to the Poincare conjecture, and was proved by Perelman.