The Periodicity Theorem, Part I
(HHR Section 9.1)

Richard Wong

eCHT Kervaire Invariant One Reading Seminar Fall 2020

Slides can be found at
http://www.ma.utexas.edu/users/richard.wong/
The Periodicity Theorem

**Theorem (9.21)**

Let $G = C_8$, and

$$D = (N_{C_2}^{C_8} \overline{\partial}_4)(N_{C_4}^{C_8} \overline{\partial}_2)(\overline{\partial}_1^{C_8}) \in \pi_{19\rho_G}^G MU((G))$$

Then multiplication by $(\Delta^G)^{16} := u_{2\rho_G}^{16} (\overline{\partial}_1^{C_8})^{32}$ gives an isomorphism

$$\pi_* (D^{-1} MU((G)))^{hG} \to \pi_* + 256 (D^{-1} MU((G)))^{hG}$$
We will use the $RO(G)$-graded slice spectral sequence to compute $\pi^G_\ast(MU^{((G))})$.

- The $E_2$ page is given by Proposition 9.7.
- Differentials are given by Theorem 9.9.

We then show that a certain class $\overline{\partial}_k u^{2k}$ is a permanent cycle (Corollary 9.13).

This implies that a class $(\Delta^G)^{2g/2}$ is a permanent cycle in the $RO(G)$-graded slice spectral sequence for $\pi^G_\ast(D^{-1}MU^{((G))})$. 

This class restricts to a unit in $\pi^u_\ast(D^{-1}MU^{((G))})$, and hence multiplication by $(\Delta^G)^{2g/2}$ gives us the Periodicity Theorem.
The $RO(G)$-graded slice spectral sequence

**Theorem**

*We have the $RO(G)$-graded slice spectral sequence for $MU^{((G))}$:*

$$E_2^{s,t} := \pi^{G}_{t-s} P_{\dim t} X \Rightarrow \pi^{G}_{t-s} MU^{((G))}$$

*with differentials $d_r : E_2^{s,t} \rightarrow E_2^{s+r,t+(r-1)}$*

Where $t \in -2m \sigma + \mathbb{Z}$
Recall that $P_0^0 MU^{((G))} \cong H\mathbb{Z}(2)$

Let $\sigma = \sigma_G$ denote the real sign representation of $G$, and recall that in Definition 3.12, we defined an element

$$u = u_{2\sigma} \in \pi_{2-2\sigma}^G H\mathbb{Z}(2) = E_2^{0,2-2\sigma}$$

Corresponding to a preferred generator of $\pi_2(H\mathbb{Z}(2) \wedge S^{2\sigma})$. We will study the elements

$$u^m \in E_2^{0,2m-2m\sigma}$$

in the $RO(G)$-graded slice spectral sequence for $\pi_\star^G(MU^{((G)})$.  

Richard Wong University of Texas at Austin

The Periodicity Theorem, Part I (HHR Section 9.1)
Consider the $\mathbb{Z} \times RO(G)$-graded ring

$$\mathbb{Z}(2)[a, f_i, u]/(2a, 2f_i)$$

with $|a| = (1, 1 - \sigma)$, $|f_i| = (i(g - 1), ig)$, and $|u| = (0, 2 - 2\sigma)$

**Proposition (9.7)**

The map

$$\mathbb{Z}(2)[a, f_i, u]/(2a, 2f_i) \rightarrow \bigoplus_{s, k \geq 0 \atop t \in \ast - k\sigma} E_{2s,t}$$

is an isomorphism in the range

$$s \geq (g - 1)((t - s) - (k - k\sigma))$$
Figure 2. The slice spectral sequence for $\pi_{-2m\sigma+\ast}^G MU^{((G))}$
Figure 3. The slice spectral sequence for $\pi^G_{-(2m+1)\sigma+*} MU^{(G)}$
The map

\[ \mathbb{Z}_2(a, f_i, u)/(2a, 2f_i) \rightarrow \bigoplus_{s,k \geq 0, t \in \ast - k\sigma} E_{2}^{s,t} \]

is given by

\[ f_i \mapsto a_i^j N_{\rho}r_i \in \pi_i^G P^i_{\rho g} MU^{(G)} \]

\[ a \mapsto a_{-\sigma} \in \pi_{-\sigma}^G P_0^0 MU^{(G)} \]

\[ u \mapsto u \in \pi_{2-2\sigma}^G P_0^0 MU^{(G)} \]
**Proposition**

*In the slice spectral sequence, we have a vanishing line*

\[ s = (g - 1)((t - s) + k\sigma) + k \]

**Proof.**

This follows by setting \( t' = \dim t \) so that \( t = t' + (k - k\sigma) \). We then have that

\[ E_2^{s,t} = \pi_{t'-s+k}^G S^{k\sigma} \wedge P_{t'} t^t MU((G)) \]

Note that \( S^{k\sigma} \wedge P_{t'} t^t MU((G)) \geq t' \), so Proposition 4.40 tells us that this group vanishes if \( t' - s + k < \lfloor t'/g \rfloor \).

Hence if \( s > (g - 1)((t - s) + k\sigma) + k \)
By the Slice Theorem (Theorem 6.1 cf Theorem 1.13), $P_{t'} MU((G))$ is contractible unless $t'$ is even.

If $t'$ is even, then $P_{t'} MU((G)) \simeq \bigvee H\mathbb{Z}(2) \wedge \hat{S}$, where $\hat{S}$ is a slice cell of dimension $t'$.

We compute $E_{s,t}^{s,t} = \pi_{t'-s+k}^G S^{k\sigma} \wedge P_{t'} MU((G))$ by considering the two cases for $\hat{S}$:

1. Either $\hat{S} = G_+ \wedge H S^\ell \rho_H$ is an induced slice cell,
2. Or $\hat{S} = S^\ell \rho_G$ is an non-induced slice cell.

We will see that in the range $s \geq (g - 1)((t - s) - (k - k\sigma))$, the homotopy groups coming from the induced slice cells vanish, and only the non-induced slice cells contribute.
Case 1

If \( \hat{S} = G_+ \wedge_H S^{\ell^\prime \rho_H} \) is an induced slice cell, then we are interested in computing \( \pi_{t' - s + k}^G \) of

\[
S^{k\sigma} \wedge H\mathbb{Z}_{(2)} \wedge G_+ \wedge_H S^{\ell^\prime \rho_H}
\]

Since the restriction of \( \sigma \) to any proper subgroup is trivial, then this is homotopic to

\[
G_+ \wedge_H (S^{k\sigma} \wedge H\mathbb{Z}_{(2)} \wedge S^{\ell^\prime \rho_H}) \cong G_+ \wedge_H (S^k \wedge H\mathbb{Z}_{(2)} \wedge S^{\ell^\prime \rho_H})
\]

Hence we are interested in computing \( \pi_{t' - s}^H (H\mathbb{Z}_{(2)} \wedge S^{\ell^\prime \rho_H}) \)
Case 1

By Proposition 4.40, \( \pi_{t'-s}^H(H\mathbb{Z}(2) \wedge S^{\ell'}\rho_H) \) vanishes if

\[
t' - s < \ell' = t'/h \quad (h = |H|)
\]

so it in particular vanishes for \( t' - s < t'/g \). Equivalently, it vanishes for

\[
s \geq (g - 1)((t - s) - (k - k\sigma))
\]
Case 2

If \( S = S^{\ell \rho_G} \) is a non-induced slice cell, then we are interested in computing

\[
\pi_j^G(S^{k\sigma} \wedge H\mathbb{Z}_2 \wedge S^{\ell \rho_G})
\]

for \( j \leq \ell + k \) and \( \ell, k \geq 0 \).

Lemma (9.1)

\[
\pi_j^G(S^{k\sigma} \wedge H\mathbb{Z}_2 \wedge S^{\ell \rho_G}) \cong
\begin{cases}
0 & \text{if } (j - \ell) < 0 \text{ or } (j - \ell) \text{ odd} \\
\mathbb{Z}/2 \cdot \{a_{\rho}^k a_{\sigma}^{k-2m} u_{2\sigma}^m\} & \text{if } (j - \ell) = 2m \geq 0 \text{ and } \ell > 0 \\
\mathbb{Z}_2 \cdot \{u_{2\sigma}^m\} & \text{if } (j - \ell) = 2m \geq 0 \text{ and } \ell = 0
\end{cases}
\]
Proof.

We write $S^{k\sigma} \wedge S^{\ell \rho_G} = S^{(k+\ell)\sigma} \wedge S^{\ell} \wedge S^{\ell(\rho_G-\sigma-1)}$ and consider the multiplication map

$$a_{\rho-\sigma}^\ell : \pi^G_j H\mathbb{Z}_2 \wedge S^{(k+\ell)\sigma} \wedge S^{\ell} \rightarrow \pi^G_j H\mathbb{Z}_2 \wedge S^{k\sigma} \wedge S^{\ell \rho_G}$$

We claim that this map is an isomorphism for $j \leq \ell + k$ and $\ell, k \geq 0$.

If $\ell = 0$, this map is an isomorphism.
Proof.

For $\ell > 0$, $S^{\ell}(\rho G - \sigma - 1)$ has one 0-cell, and all other $G$-cells are induced and in positive dimension.

Since the restriction of $\sigma$ to every proper subgroup is trivial, it follows that to obtain $S^{k\sigma} \wedge S^{\ell\rho G}$ from $S^{(k+\ell)\sigma} \wedge S^{\ell}$, one attaches induced $G$-cells of dimension greater than $(k + 2\ell)$.

Hence $a^{\ell}_{\rho - \sigma}$ is an isomorphism for $j < k + 2\ell$, and hence for $j \leq \ell + k$ since $\ell > 0$.

Therefore, in our range, we are interested in computing

$$\pi_j^G(H\mathbb{Z}_2(2) \wedge S^{(k+\ell)\sigma} \wedge S^\ell)$$

Which was done in Proposition 3.16.
It remains to identify the summand of non-induced slice cells in $MU^{((G))}$. That is, we need the algebra structure as well.

Recall that we have an associative algebra equivalence

$$\bigvee_{k \in \mathbb{Z}} P_k^k MU^{((G))} \cong H\mathbb{Z}(2) \wedge S^0[G \cdot \bar{r}_1, \cdots]$$

We can do so by identifying the summand of non-induced slice cells in each $S^0[G \cdot \bar{r}_i]$ and smashing them together.

**Proposition**

*The associative algebra map*

$$S^0[N\bar{r}_1, \cdots] \to S^0[G \cdot \bar{r}_1, \cdots]$$

*is the inclusion of the summand of non-induced slice cells.*
Proof.

Take the generating inclusion $\bar{r}_i : S^{i\rho C_2} \to S^0[\bar{r}_i]$

We then apply the norm $N^G_{C_2}$ to obtain $N\bar{r}_i : S^{i\rho G} \to S^0[G \cdot \bar{r}_i]$. We can then extend it to an associative algebra map $S^0[N\bar{r}_i] \to S^0[G \cdot \bar{r}_i]$, which we claim is the inclusion of the summand of non-induced slice cells.

Recall that

$$S^0[G \cdot \bar{r}_i] \simeq \bigvee_{f : G/C_2 \to \mathbb{N}_0} S^{V_f}$$
Proof.

\[ S^0[G \cdot r_i] \simeq \bigvee_{f: G/C_2 \to \mathbb{N}_0} S^{V_f} \]

Decompose the right hand side over the \( G \)-orbits.

Since an indexed wedge over a \( G \)-orbit is induced from the stabilizer of any element of the orbit, the summand of non-induced slice cells consists of those \( f \) which are constant.

If \( f \) is the constant function \( n \), then \( V_f = n\rho_G \), hence the summand of non-induced slice cells is

\[ \bigvee_{n} S^{n\rho_G} \]
Consider the $\mathbb{Z} \times RO(G)$-graded ring
\[
\mathbb{Z}(2)[a, f_i, u]/(2a, 2f_i)
\]
with $|a| = (1, 1 - \sigma)$, $|f_i| = (i(g - 1), ig)$, and $|u| = (0, 2 - 2\sigma)$.

**Proposition (9.7)**

The map
\[
\mathbb{Z}(2)[a, f_i, u]/(2a, 2f_i) \rightarrow \bigoplus_{s,k \geq 0, t \in \ast - k\sigma} E_{s,t}^2
\]
is an isomorphism in the range
\[
s \geq (g - 1)((t - s) - (k - k\sigma))
\]
\textbf{Figure 2.} The slice spectral sequence for $\pi_{-2m\sigma+*}^G MU((G))$
Figure 3. The slice spectral sequence for $\pi^G_{-(2m+1)\sigma+*} MU^{(G)}$.
By construction, the $f_i$ represent the elements $f_i = a^k \rho G N \tau_i \in \pi_i^G MU^{((G))}$, and are hence permanent cycles.

Similarly, $a$ represents the element $a_\sigma \in \pi_{-\Theta}^G MU^{((G))}$, and also is a permanent cycle.

**Theorem (9.9)**

*In the slice spectral sequence for $\pi_*^G (MU^{((G))})$, the differentials $d_i(u^{2^{k-1}})$ are zero for $i < r := 1 + (2^k - 1)g$, and

$$d_r(u^{2^{k-1}}) = a^{2^k} f_{2^k-1}$$*
Note that on the vanishing line

\[ s = (g - 1)((t - s) + k\sigma) + k \]

is the algebra

\[ \mathbb{Z}_2[a, f_i]/(2a, 2f_i) \]

Recall that in Proposition 5.50, the kernel of the map

\[ \Phi^G : H\mathbb{Z}_2[N\bar{r}_1, \cdots] \to \pi_*^G \Phi^G MU((G)) = \pi_* MO[a_{\sigma}^{\pm 1}] \]

is the ideal \((2, f_1, f_3, f_7, \cdots)\)

Hence any non-trivial differentials into the vanishing line must land in this idea.
Proof.

We prove the Slice differential theorem by induction on \( k \). Assume the result for \( k' < k \).

In the range \( s \geq (g-1)(t-s-(k-k\sigma)) \), after resolving the induction differentials, there are two modules over \( \mathbb{Z}_2[f_i]/(2f_i) \):

one generated by \( a^k \), which is free over the quotient ring

\[
\mathbb{Z}/2[f_i]/(f_1, f_3, \ldots, f_{2^{k-1}-1})
\]

and one generated by \( u^{2^{k-1}} \).

Since the differential must land in \( (2, f_1, f_3, f_7, \cdots) \), for degree reasons, the only possible differential on \( u^{2^{k-1}} \) is the one asserted by the theorem. We must show that \( u^{2^{k-1}} \) does not survive the spectral sequence.
Proof.

It suffices to do so after inverting $a$. Recall that for $G = C_{2n}$, up to fibrant replacement,

$$\pi_* \Phi^G(X) = \pi_* (\tilde{E}P \wedge X) \cong a_{\sigma}^{-1} \pi_*^G X$$

Inverting $a$ on the map $\pi_*^G MU((G)) \to \pi_*^G H\mathbb{Z}_2(2)$ yields the map

$$\pi_* \Phi^G(MU((G))) = \pi_* MO \to \pi_* \Phi^G(H\mathbb{Z}_2(2))$$

By Proposition 7.6, this map is 0 in positive degrees. However, if $u^{2^{k-1}}$ is a permanent cycle, so is $a^{-2^k} u^{2^{k-1}}$, but this would represent a class $b^{2^{k-1}}$ in $\pi_* \Phi^G(H\mathbb{Z}_2(2)) \cong \mathbb{Z}/2[b]$, which is a contradiction. $\square$
Permanent cycles

Write

$$\overline{\partial}_k = N\bar{r}_{2^k-1} \in \pi_{(2^k-1)\rho\mu}^G\mu^{((G))}$$

Note that $f_{2^k-1} = a_{\rho}^{2^k-1} \overline{\partial}_k$.

Also observe that we have the identity

$$f_{2^{k+1}-1} \overline{\partial}_k = a_{\rho}^{2^{k+1}-1} \overline{\partial}_{k+1} \overline{\partial}_k = f_{2^k-1} a_{\rho}^{2^k} \overline{\partial}_{k+1}$$

$\overline{\partial}_k$ is represented in the $RO(G)$-graded slice spectral sequence by an element also denoted $\overline{\partial}_k \in \pi_{(2^k-1)\rho\mu}^G\mu^{((G))}$. 

Richard Wong
University of Texas at Austin
The Periodicity Theorem, Part I (HHR Section 9.1)
Corollary (9.13)

In the RO(G)-graded slice spectral sequence for MU((G)), the class \( \partial_k u^{2^k} \) is a permanent cycle.

Proof.

Set \( r = 1 + (2^{k+1} - 1) g \). By the Slice differential theorem, the differentials \( d_i(\partial_k u^{2^k}) = \partial_k d_i(u^{2^k}) \) are zero for \( i < r \). Moreover,

\[
d_r(\partial_k u^{2^k}) = \partial_k a^{2^{k+1}} f_{2^{k+1}-1} = a^{2^{k+1}} f_{2^k - 1} a^{2^k}_p \partial_k
\]

However, setting \( r' = 1 + (2^k - 1) g \), note that \( r' < r \). We also have

\[
d_{r'}(u^{2^{k-1}} a^{2^k} a^{2^k}_p \partial_{k+1}) = a^{2^k} f_{2^k - 1} a^{2^k}_p \partial_{k+1}
\]

Therefore, we actually have that \( d_r(\partial_k u^{2^k}) = 0 \).
Proof.

It remains to show that the higher differentials vanish. We show that they land in a region that is 0 in the $E_2$ term.

Note that $\partial_k u_k^{2^k} \in \pi^G_{2^k(2-2\sigma)+(2^k-1)\rho_G} P^{(2^k-1)\rho} MU((G))$

The differential $d_{i+1}$ decreases $t-s$ degree by 1, and increases $t$ degree by $i$. Hence we wish to study

$$\pi^G_{2^k(2-2\sigma)+(2^k-1)\rho_G-1} P^{(2^k-1)\rho+i} MU((G))$$

for $i + 1 > r = 1 + (2^{k+1} - 1)g$. 

Richard Wong  
University of Texas at Austin  
The Periodicity Theorem, Part I (HHR Section 9.1)
The RO(G)-graded slice spectral sequence

The Slice Differentials Theorem

---

**Proof.**

Equivalently, we wish to study

$$\pi^G_{2k+1-1}(S^{2k+1}\sigma \wedge S^{-(2^k-1)}\rho \wedge P^{((2^k-1)g+i)} \wedge MU((G)))$$

We rewrite this as

$$\pi^G_{2k+1-1}(S^{2k+1}\sigma \wedge X_i)$$

Note that $X_i \geq i$, so by Proposition 4.40, $\pi^G_j X_i = 0$ for $j < \lfloor i/g \rfloor$.

Since $S^{2k+1}\sigma$ is (-1)-connected, then $\pi^G_{2k+1-1}(S^{2k+1}\sigma \wedge X_i)$ vanishes for $i \geq 2^{k+1}g$. 

---
Proof.

For the remaining values of \( i \), since they are strictly between \( 2^{k+1}g \) and \((2^{k+1} - 1)g\), then \( i \) is not divisible by \( g \). Since \( MU((G)) \) is pure, then \( P^{(2^{k-1})g+i}(MU((G))) \) is induced from a proper subgroup. Therefore, so is \( X_i \).

We therefore have an equivalence

\[
S^{2k+1} \wedge X_i \simeq S^{2^{k+1}} \wedge X_i
\]

Therefore, we have that

\[
\pi^{G}_{2k+1-1}(S^{2k+1} \wedge X_i) = \pi^{G}_{2k+1-1}(S^{2^{k+1}} \wedge X_i) = 0
\]

since \( X_i \geq 0 \).