Reflections on Authentic Assessment

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Slides available at rwongmath.github.io

Doing mathematics
is nothing like
taking a timed,
closed-book exam.

Taking Exams:

- Solve a bunch of problems,
    - under time pressure.
- Done individually.
- Limited (or no) resources.
- “Show your work”.

Doing math:

- Understand one problem/concept deeply,
    - by some deadline.
- Collaborative.
- Can use any/all resources.
- Communicate and explain your ideas.

Can we design assignments that reflect how mathematicians practice mathematics?

Outline:

- What is a "**Challenge Problem Report**"?
- How do I **implement** these assignments?
  - How do I **prepare my students** for this assignment?
  - How do I **grade/give feedback** on these assignments?
- Final thoughts/reflections.

What is a “Challenge Problem Report?”

Challenge Problem Reports are **technical writing assignments**, where you will apply mathematics to more challenging (real world) problems.

These challenge reports are designed to both help you **develop both your mathematical reasoning and your communication skills**. Accordingly, you will be graded on both the *correctness* of the mathematics, as well as *how you explain* your mathematical ideas.

32A CHALLENGE PROBLEM REPORT 2

FALL 2023

Abstract. In this Challenge Problem Report, you will study the Korteweg–De Vries (KdV) equation, which is a partial differential equation.

1. Partial differential equations

Given a function of n variables $f(x_1, \cdots, x_n) : \mathbb{R}^n \to \mathbb{R}$, we have learned in class how to compute the partial derivatives $\frac{\partial f}{\partial x_i}$. For convenience, we will assume for convenience that our multivariable functions $f : \mathbb{R}^n \to \mathbb{R}$ are **smooth** (that is **all** higher partial derivatives of f exist and are continuous).

However, in real life, it is instead often the case that the opposite scenario occurs. That is, through measurements and/or experimental data, we begin with information about the partial derivatives $\frac{\partial f}{\partial x_i}$, and we wish to find a multivariable function f that satisfies our observations.

For example, our data might lead us to observe that there might be some equation that describes the relationship between various partial derivatives. Such an equation is called a **partial differential equation** (PDE).

Unfortunately, it turns out to be extremely difficult to find explicit solutions to PDEs. In fact, studying PDEs is a large and active field in both pure and applied mathematical research. For example, there is a lot of interest in methods to numerically approximate solutions to PDEs. There is also a lot of interest in studying and characterizing solutions to particularly important PDEs, such as the Navier–Stokes equations.

2. The KdV equation

In this challenge problem report, we will study a specific PDE, called the **Korteweg–De Vries (KdV) equation**. This PDE describes the motion of waves in shallow water.

Given a function of two variables $u(x,t) : \mathbb{R}^2 \to \mathbb{R}$, we will think of the function $u(x,t)$ as describing the height of a wave (in meters) at some coordinate $x \in \mathbb{R}$ (described in meters) at time t (described in seconds). That is, the variable x describes a position, and the variable t describes time.

We can describe the motion of the wave using a partial differential equation:

Definition 2.1. Given a function of two variables $u(x,t) : \mathbb{R}^2 \to \mathbb{R}$, the (one-dimensional) **KdV equation** is a partial differential equation of the form

$$u_t + \alpha u_{xxx} + \beta u u_x = 0$$

for given constants $\alpha, \beta \in \mathbb{R}$.

It turns out that one can find explicit solutions to this PDE - you will study a particular solution in the exercises below.

To complete the second challenge problem report, you will write up solutions to the following problems. Your write-up should include exposition and read like a chapter or section of a textbook. Be sure to clearly label your answers to the questions.

(1) Consider the function $u(x,t) = \operatorname{sech}^2(x-t)$.
  (a) Show that $u(x,t)$ satisfies the KdV equation
  $$u_t + \frac{1}{4}u_{xxx} + 3uu_x = 0$$
  (b) Sketch the trace of the graph $z = u(x,t)$ in the planes $t = 0$, $t = 1$, and $t = 10$.

(2) Find a function that describes the coordinate of the peak of the wave at a fixed time t.
  (a) How fast, and in what direction, does the peak of the wave travel?

(3) Suppose that you would instead like to describe the motion of the wave that satisfies the following conditions:
  - the height of the wave is 2 meters tall.
  - at the time $t = 0$, the coordinate of the peak of the wave is at $x = 1$.
  - the wave travels at a constant speed 5 meters per second in the positive x direction.
  - the width of the wave and the height of the water is the same as in question (1).

  (a) Modify the function $\operatorname{sech}^2(x-t)$ to obtain a function $f(x,t)$ that describes the wave under the above conditions.
  (b) Determine the KdV equation that your function $f(x,t)$ satisfies. (That is, find the appropriate coefficients α and β).

3. Bonus Food for Thought

These questions are optional, and do not need to be answered for full credit.

The solution to the KdV equation studied in the challenge report describes the motion of a solitary wave. One might wonder if it possible to describe the travel of multiple waves at once (e.g. an n-soliton solution) [DJ89].

It turns out that the following function can describe the travel of 2 waves at once (and how they interact)

$$u(x,t) = 12\left(\frac{3 + 4\cosh(2x - 8t) + \cosh(4x - 64t))}{[3\cosh(x - 28t) + \cosh(3x - 36t)]^2}\right) \tag{3.1}$$

(A) Study the equation 3.1:
  (a) Determine the KdV equation that $u(x,t)$ satisfies.
  (b) Sketch t-traces of the graph $z = u(x,t)$ in various planes.
  (c) How fast, and in what direction, do the two waves travel?

(B) The waves described by equation 3.1 merge and form a single peak for a certain t-value.
  (a) Find the t-value where there is a single peak.
  (b) Modify equation 3.1 so that there is always a double peak.

(C) Give a general equation for the 2-soliton solution in terms of the two speeds of the waves.

(D) Give a general equation for the n-soliton solution.

References

[DJ89] P. G. Drazin and R. S. Johnson. *Solitons: An Introduction*. 2nd ed. Cambridge Texts in Applied Mathematics. Cambridge University Press, 1989. DOI: 10.1017/CBO9781139172059.

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Logistics:

- The students begin working on the assignment **in small groups** during discussion.
- The students have roughly a week to submit a **polished write up** of their solutions.

Assignment instructions:

Your write-up should read like a chapter or section of a textbook, and stand on its own. That is, you should

- motivate the problem,
- define the terms you use, and
- you should clearly state any assumptions, equations, or theorems you use.

In other words, you should be able to hand your solutions to any student taking this course, and they should be able to understand both the context of the problem and your solutions. I highly recommend using **peer review** as a way to improve your solutions!

Assignment instructions, continued:

Your submission should

- Include an introduction that explains the context and motivation for the problems.
- Clearly justify and explain your solutions using words and mathematical reasoning.
- Clearly label your answers.
- Include a list of your group/collaborators.
- Include citations for any outside references you use.

Sample student work:

Suppose that you would instead like to describe the motion of the wave that satisfies the following conditions:

- the height of the wave is 2 meters tall.
- at the time $t = 0$, the coordinate of the peak of the wave is at $x = 1$.
- the wave travels at a constant speed 5 meters per second in the positive x direction.
- the width of the wave and the height of the water is the same as in question (1).

3.1 Part (a): Question

Modify the function $\operatorname{sech}^2(x - t)$ to obtain a function $f(x, t)$ that describes the wave under the above conditions.

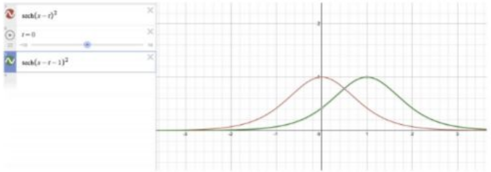
3.2 Part (a): Solution

To obtain our new function $f(x, t)$, we will start by considering a general form of the function $\operatorname{sech}^2(x - t)$.

$$f(x,t) = A\operatorname{sech}^2(a(x - bt - c)) + d$$

The constants A, a, b, c, and d represent all the possible transformations of our original function $u(x, t)$.

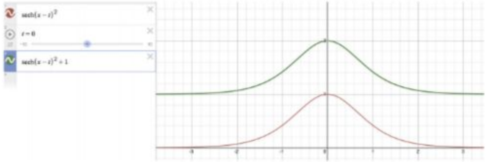
For instance, a change in c would result in a horizontal translation of the original function, as demonstrated by the following graph.



The second condition states that the coordinate of the peak of the wave is at $x = 1$ when time $t = 0$. This statement is asking for a horizontal shift of 1 unit in the x-direction. Therefore, $c = 1$.

$$f(x,t) = A\operatorname{sech}^2(a(x - bt - 1)) + d$$

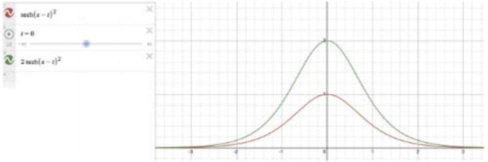
Furthermore, a change in d would result in a vertical translation of the original function, as shown in the following graph.



Part of the fourth condition states that the height of the water is the same as in $u(x, t)$. In question (1), the height of the water is 0, as there was no vertical translation in the initial function $u(x, t)$. Therefore, our new function will also have no vertical translation, meaning $d = 0$.

$$f(x,t) = A\operatorname{sech}^2(a(x - bt - 1))$$

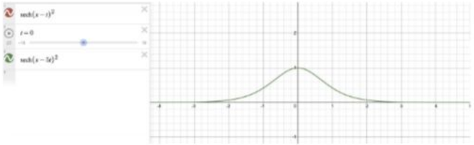
Next, we know that if we multiply the function by some constant A, we are scaling the entire function by some constant A. This is shown in the following graph.

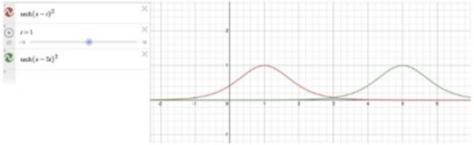


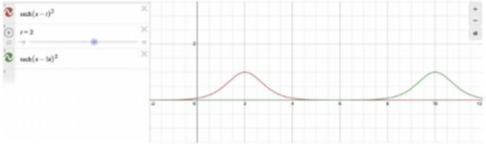
The first condition asks for the height of the wave to be 2 meters tall. We know that the output of our function $f(x, t)$ will give us the height of the wave. In $u(x, t)$, the height of the wave was 1 meter tall, since the constant $A = 1$. If we multiply our function by 2, we will scale the height of wave to 2 meters. Therefore, $A = 2$.

$$f(x,t) = 2\operatorname{sech}^2(a(x - bt - 1))$$

Next, we will satisfy condition 3. We know that the variable t represents time and the variable x represents position. If we multiply t by some constant b, that demonstrates that our position x must be moving b times as fast in the same amount of time t as the original function. In other words, for every unit increase in time t of the new function $f(x,t)$, the position x of $f(x,t)$ will increase b times as much as the original function $u(x,t)$. We can see that as t is increasing in the following few graphs, f will travel 5 times as far as u will.



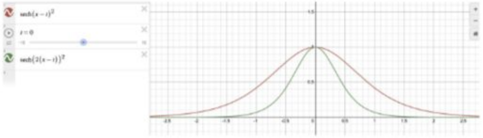




This indicates that the constant b dictates the speed of the wave, so $b = 5$.

$$f(x,t) = 2\,\mathrm{sech}^2(a(x - 5t - 1))$$

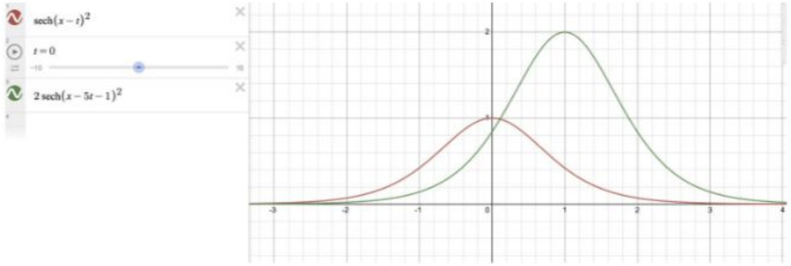
Finally, we need to ensure that the width of the wave is the same as in question (1). as stated in condition 4. The constant a represents how much we are scaling the width of the wave by, which we can see in the following graph.



In question (1), the constant $a = 1$ in $u(x,t)$. Therefore, in our modified equation, $a = 1$.

Our final, modified equation is:

$$f(x,t) = 2\,\mathrm{sech}^2(x - 5t - 1)$$



These assignments encourage students to learn collaboratively:

These assignments get students thinking deeply about the material.

- “How can we explain the changes we make to the function sech(x-t)?”
- “What does it mean that the width of the wave doesn’t change?”
- “Wait, why does adding a constant change the water level?”

These assignments allow students to explore mathematics.

Plugging these values into Δv yields

$$\begin{aligned} v_{xx} + v_{yy} &= 0 \\ 2a + 2c &= 0 \\ c &= -a \end{aligned}$$

In order for our homogeneous polynomial to satisfy the Laplace equation, there must be a relationship between the coefficients a and c such that $c = -a$. Replace c in our original expression to get

$$v(x, y) = ax^2 + bxy - ay^2 \quad a, b \in \mathbb{R},$$

the general function for harmonic homogeneous polynomials of degree two.

Now that we have found a general equation for harmonic polynomials of degree two, can we find a general equation for polynomials of degree 4? Degree 5? Degree n? What about for functions that have three variables instead of two? How about k variables?

In fact, there is a generalized equation for homogeneous polynomials of k variables and degree n. However, proving the existence of such a equation proved to be way outside my skill set. Instead, here is a link to a journal article by Miles and Williams regarding the topic: `https://doi.org/10.2307/2032337`.

How do I implement these assignments?

Preparing students for group work:

Create an **inclusive** and **equitable** learning environment where *all* students feel welcome to voice their questions and ideas.

"Everyone can have joyful, meaningful, and empowering mathematical experiences." - Federico Ardila

Creating an inclusive and collaborative learning environment:

- **Set the tone** on day one.
- **Encourage questions** and participation during lecture.
- Use **low-stakes, group quizzes** and encourage students to communicate and explain their ideas.

"The only way to learn mathematics is to do mathematics." - Paul Halmos

Framing these assignments:

These assignments are also designed to **simulate how mathematicians think about and do mathematics.**

In other words, they are an opportunity for you to **demonstrate your understanding and mastery of the material** on something other than a timed exam.

How do I grade these assignments?

Grading these assignments:

Graded on **three major categories**:

- ❖ **Readability and Fluency**
  - ➢ How well do students communicate their ideas and explain their solutions?
- ❖ **General Strategy**
  - ➢ Do the students demonstrate an understanding of the relevant concepts?
- ❖ **Details**
  - ➢ Do the students accurately/correctly solve the problems?

Readability and Fluency

- **(E)** Excellent/Exemplary:
  - The solutions are written clearly and concisely, and follow a logical and well-organized manner.
  - The solutions demonstrate mathematical fluency, and precisely states any definitions, theorems, or results.
  - The exposition is well-motivated and insightful, going above and beyond solving the problems.
- **(M)** Meets Expectations:
  - The solutions are written clearly and concisely, and follow a logical manner.
  - The solutions demonstrate mathematical fluency, and precisely states any definitions, theorems, or results.

    The solutions provide exposition/motivation of the problem.
- **(R)** Revision Needed:
  - The solutions are organized in a logical manner.
  - However, the solutions lacks clarity, conciseness, precision, or mathematical fluency, and/or lacks exposition.
- **(N)** Not Assessable:
  - The solutions are difficult to read and/or understand.
  - The solutions fail to be organized in a logical manner; and they may further lack clarity, conciseness, precision, or mathematical fluency.
  - There is no effort put into exposition.

General Strategy

- **(E)** Excellent/Exemplary
  - The solutions demonstrate a deep understanding of the problems, as well as mastery of the mathematical tools needed to solve the problems.
  - The general strategy and methods taken towards solving the problems is sound, and will lead to a complete solution of the problems.
- **(M)** Meets Expectations:
  - The solutions demonstrate an understanding of the problems, as well as the mathematical tools needed to solve the problems.
  - The general strategy taken towards solving the problems is sound, and will lead to a solution of the problems (with minor gaps or flaws).
- **(R)** Revision Needed:
  - The solutions demonstrate some confusions or misconceptions about the problems and/or the mathematical tools needed to solve the problems (e.g. attempting to use theorems or results that are not relevant).
  - The general strategy is flawed, and may only lead to a partial solution.
- **(N)** Not Assessable:
  - The solutions demonstrate major confusions or misconceptions about the problems, and/or the mathematical tools needed to solve the problems.
  - There may be no coherent general strategy to solving the problem.

Details

- **(E)** Excellent/Exemplary:
  - The solutions provide a full and complete explanation of the methods used to solve the problem.
  - There are no flaws or gaps in logic.
- **(M)** Meets Expectations:
  - The solutions provide a substantive explanation of the methods used to solve the problem. (e.g. the solutions generally apply theorems appropriately; justify and/or prove any intermediary results).
  - There may be minor flaws or minor gaps in logic.
- **(A)** Approaches Expectations
  - The solutions provide some explanations of the methods used to solve the problem. (e.g. the solutions generally apply theorems appropriately; but might not justify or prove any intermediary results).
  - There may be minor flaws or minor gaps in logic.
- **(R)** Revision Needed:
  - The solutions provide partial or incorrect explanations of the methods used to solve the problem. (e.g. the solutions occasionally apply theorems correctly; the solutions may fail to verify key hypotheses; the solutions may lack key details or intermediary results).
  - There may be major flaws or gaps in logic.
- **(N)** Not Assessable:
  - The solutions do not provide any logical explanations or justifications of the methods used to solve the problem.
  - There may be several major flaws or gaps in logic.

Converting the rubric to grades:

- There are multiple ways to earn a certain grade.
- The assignment must be reasonably attempted to earn a grade.
- A submission should earn 100% if no **significant** revisions are necessary.

Percentage Grade	Additional Notes:	Readability and Fluency	General Strategy	Details
100%	Must have at least one Excellent/Exemplary	≥M	≥M	≥M
92%		M	M	M
89%	Must exceed the standards for 86% in at least one category (**E/E/A**, **E/M/A**, **M/M/A**, or **R/M/M**)	≥R	≥M	≥A
86%		R	M	A
82%	Also includes **E/R/A** and **M/R/A**	≥R	R	A
79%	Must exceed the standards for 76% in at least one category (**E/E/R**, **E/M/R**, **M/E/R**, **M/M/R**, **E/R/R**, **M/R/R**, **R/E/R**, or **R/M/R**)	≥R	≥R	R
76%		R	R	R
72%		N	R	R
69%		N	R	N
59%	Assignment must be reasonably attempted.	N	N	N

What is it like grading these assignments?

- Use **Gradescope** to create a general feedback rubric
  - For large classes, resist the urge to give line-by-line feedback!
- Do two passes:
  - First, read **details** carefully.
  - Second, assess the overall **strategy/readability and fluency**.

⚙ Select many ✖

▾1 Common errors

+0.0

Solutions lack an introduction that explains the context and motivation for the problems. Writing the answers is not enough!

+0.0

The motivation/exposition should be done in your own words.

+0.0

The solutions should not assume that the reader knows what the questions are.

+0.0

Solutions lack sufficient explanation and/or justification.

+0.0

You should explain your solutions step-by-step, rather than writing a block of text, followed by a bunch of computations!

+0.0

Solutions are difficult to follow and/or understand.

+0.0

Exposition is too verbose - it is not clear what the important ideas are.

+ Add Rubric Item to Group

⚙ Select many ✖

▾1 Common errors

+0.0

(1a) Error in computing u_t

+0.0

(1a) Error in computing u_x, u_{xx}, or u_{xxx}

+0.0

(1) Did not explain key steps in verifying u(x,t) satisfies KdV equation (e.g. hyperbolic identities).

+0.0

(1b) Errors in graphing traces (the graphs do not intersect x-axis).

+0.0

(2) Did not mathematically determine the peak of the wave (either by optimization or by observing it is a translation).

+0.0

(2) Invalid or insufficient justification for finding peak of the wave (either through optimization or translation).

+0.0

Error in describing the peak of the wave

+0.0

(2a) Did not mathematically determine the speed of the wave.

+0.0

(3a) Did not provide sufficient justifications for modifications of $sech^2(x-t)$ (e.g. graphs, translations, derivatives, etc.)

How do I integrate the learning cycle into these assignments?

The revision process:

I offer an **optional** revision assignment for the **first** challenge report.

- ❖ This assignment has 2 parts:
  - ➢ answering a reflection assignment, AND
  - ➢ submitting a revised challenge problem report.
- ❖ The grade on the revised assignment **replaces** the original Challenge Problem Report 1 grade.

The reflection assignment:

- In your own words, **what did you do well in your first draft?**
- In your own words, **what did you need to improve** on in your revised solutions?
  - You should also describe the changes you made in your revised solutions.
- **What grade do you think your revised solutions will earn?**
  - You should assign your revised solutions a grade using the grading rubric at the bottom of the page.
- **What will you do differently for the second challenge problem report?**

Summary:

- A “**Challenge Problem Report**” is a technical writing assignment that engages students in doing mathematics.
- I prepare my students for this assignment by **creating an inclusive and collaborative learning environment**
- I grade/give feedback on these assignments in terms of **Readability & Fluency**, **General Strategy**, and **Details.**