# Surfaces, Space, and Hyperspace

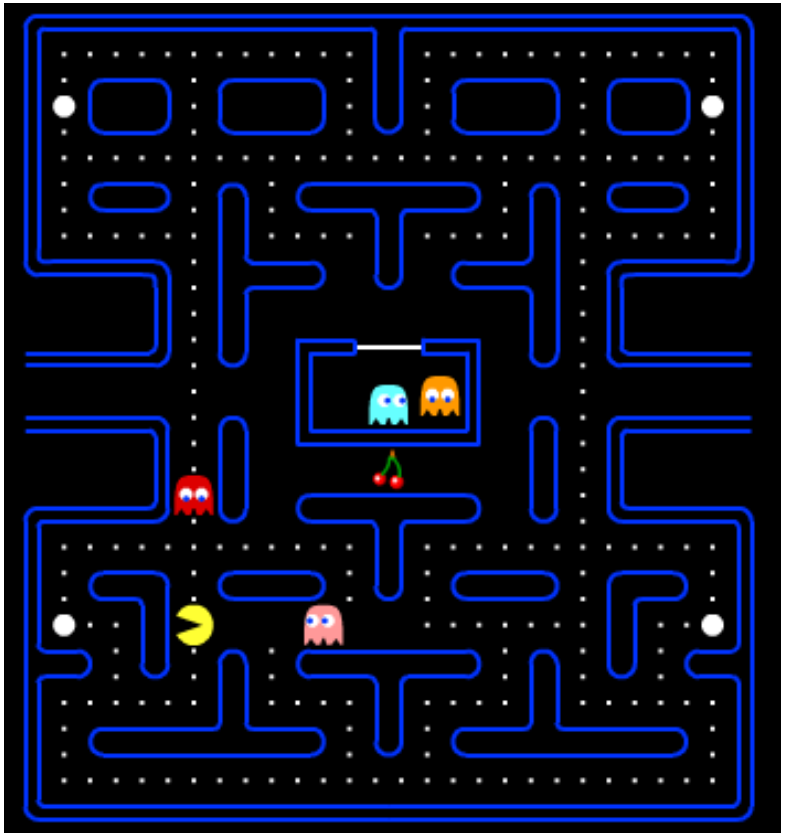
## An exploration of 2, 3, and higher dimensions

Richard Wong

UT Austin SMMG Talk, Feb 2018

Slides can be found at
http://www.ma.utexas.edu/users/richard.wong/Notes

# Pac-Man



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- So even though Pac-man's world looks flat, it's actually not flat at all!
- When we glue together the sides, we see that Pac-man's world is the surface of a **cylinder**.

# Asteroids


Another classic arcade game.

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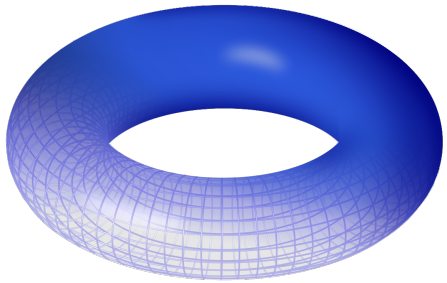

A torus. (Source: Wikipedia)

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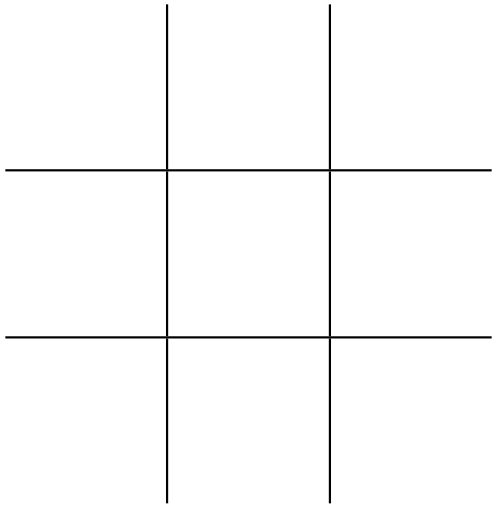

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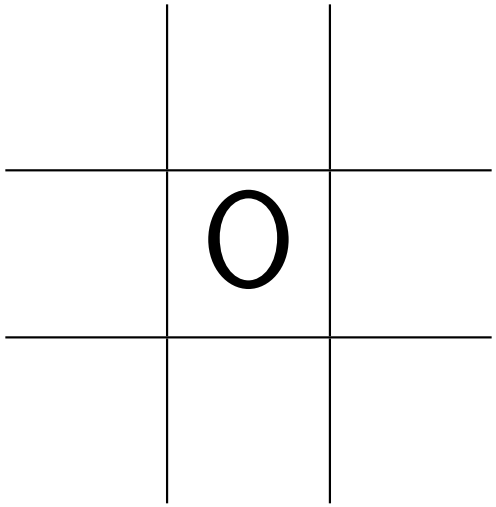

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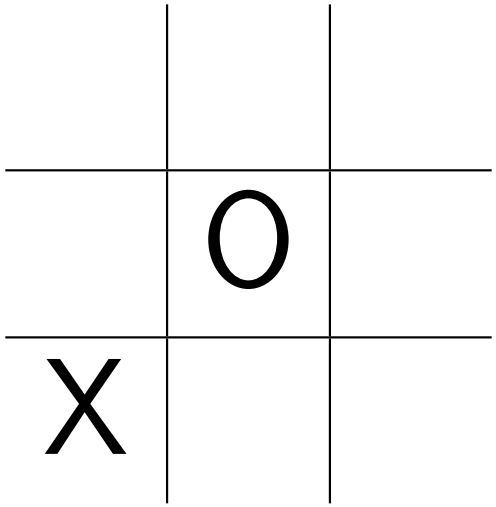

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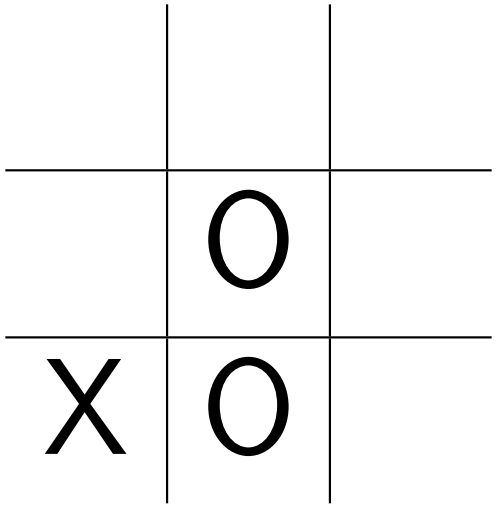

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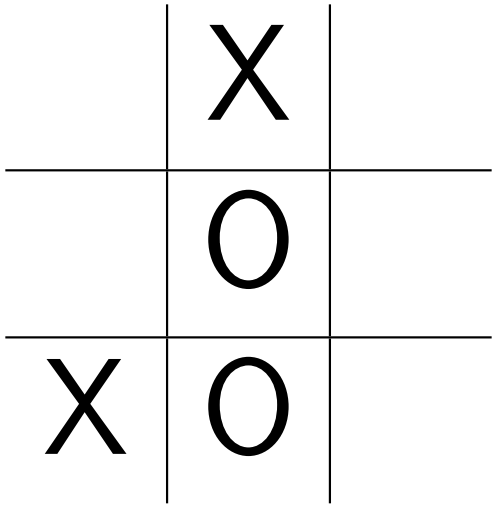

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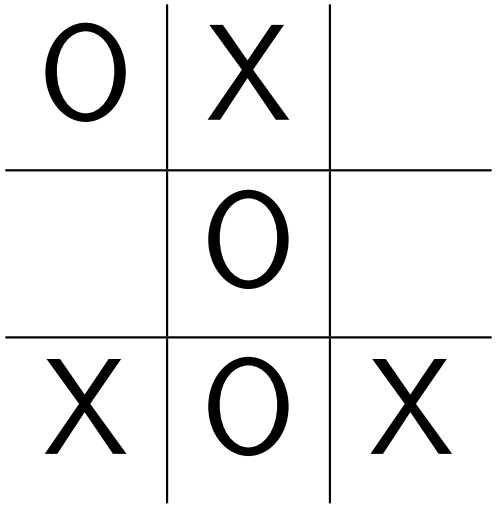

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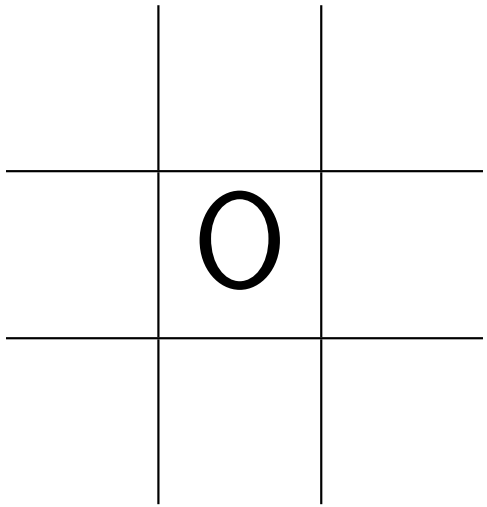

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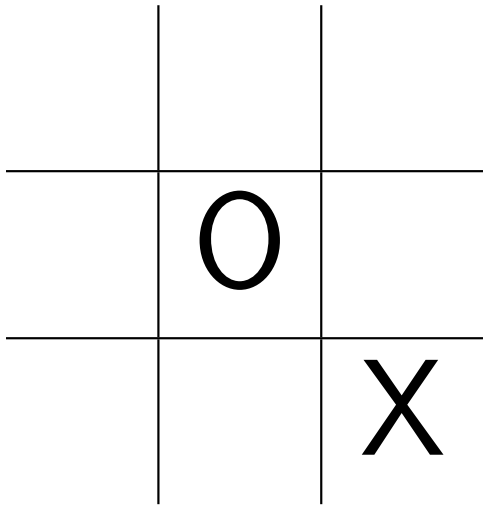

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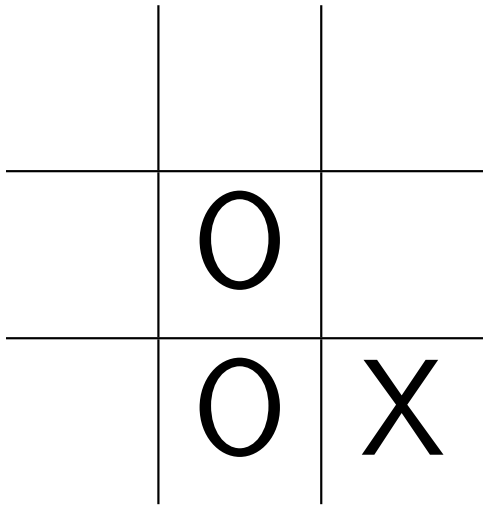

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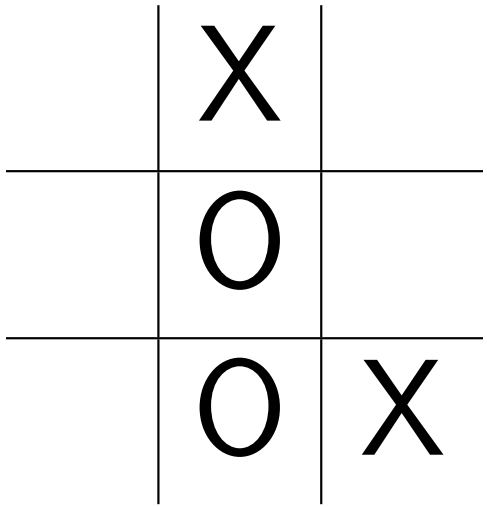

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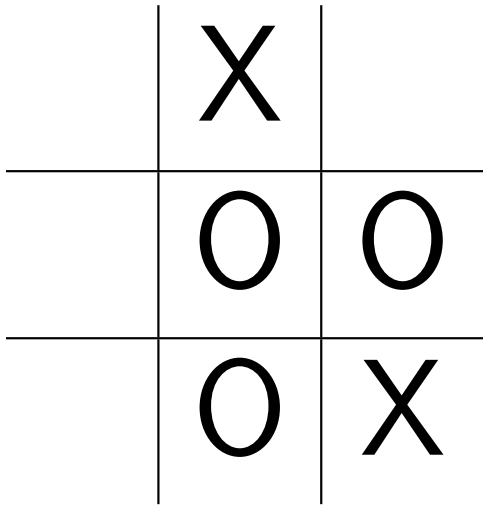

# Tic-Tac-Toe

What happens if we play tic-tac-toe on a torus?

| | | |
|---|---|---|
| | X | |
| X | O | O |
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# Tic-Tac-Toe

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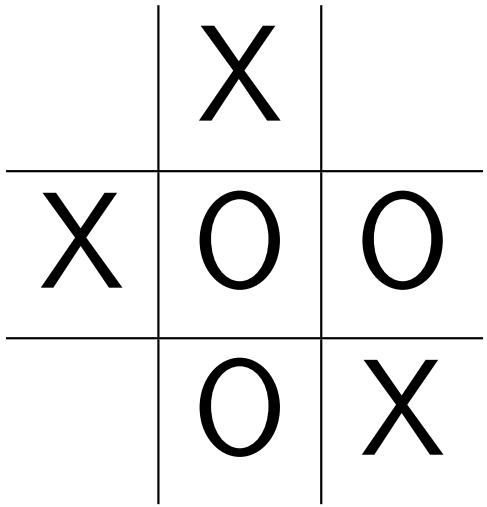


**Challenge:** Solve (determine the optimal gameplay for) cylindrical and toroidal tic-tac-toe.

# Chess

What happens if we play chess on a cylinder?

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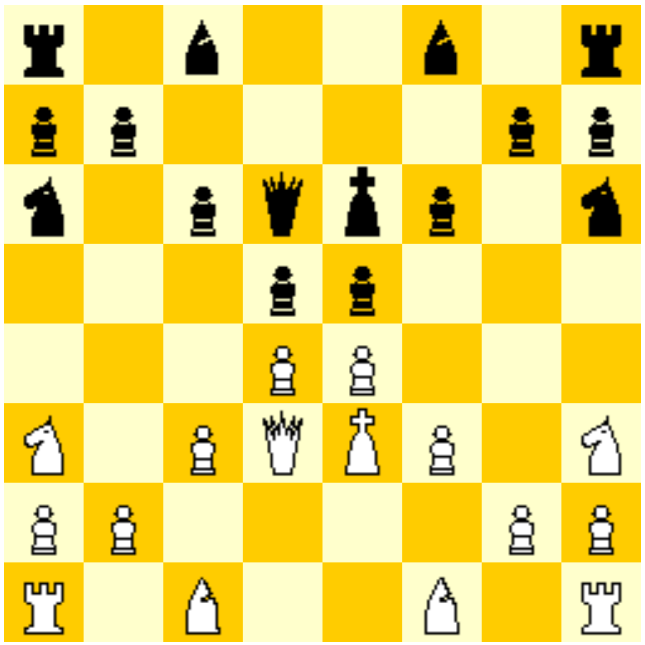


Torus chess variant by Karl Fischer.

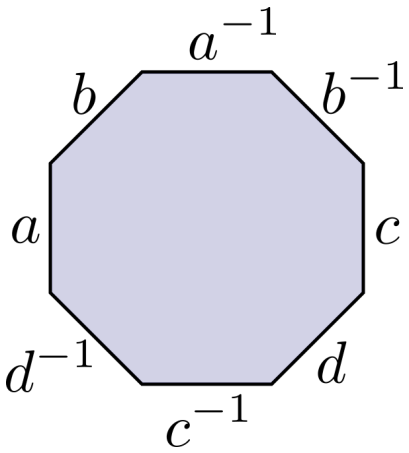
# The Rules

Start with a **convex polygon** and label edges such that edge label appears twice, once in the clockwise direction, and once in the counterclockwise direction.

- You can rotate and flip your paper over.
- If you have two adjacent edges with the same label, you can cancel them. In other words, redraw your polygon without those two edges.
- You can draw a new line between two corners and label it. You can then cut along this line.
- You can glue together identified edges.

# Questions

1. Can you construct a sphere?
2. What do you get from the following diagram?




A polygon. (Source.)

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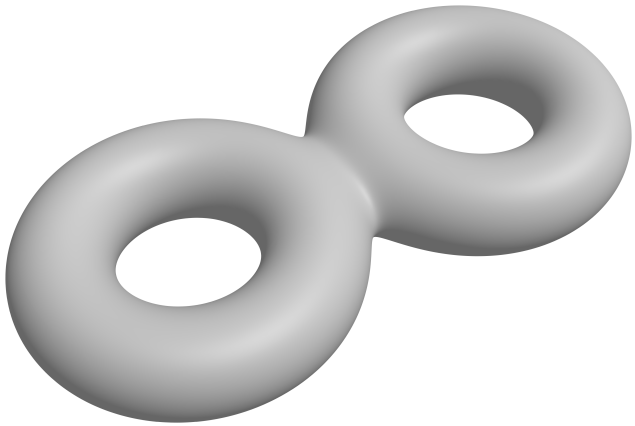

A double torus.

# Surfaces

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- It turns out that via cutting and pasting, you can classify mathematical objects called **surfaces**. These are objects that locally look like they're flat, like the sphere or torus.
- In particular, we can classify surfaces that are **closed** and **orientable**.
- A **closed** surface is a surface without a boundary edge. The cylinder is not a closed surface, while the torus is.
- An **orientable** surface is one that you can have a consistent compass orientation. We will see a non-example shortly.

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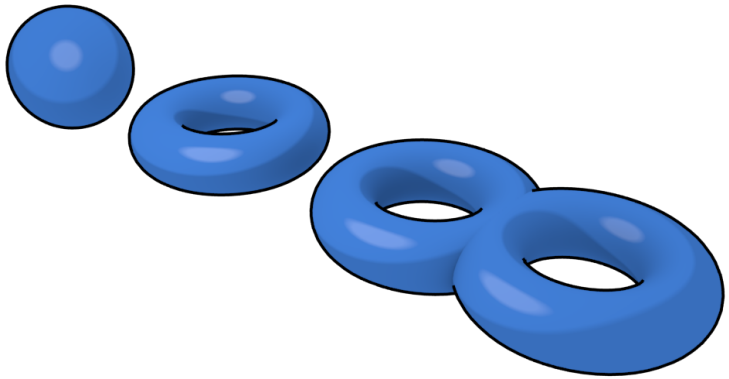
- There are countably many surfaces distinguished by **genus**, which is the number of holes.



The closed orientable surfaces. (Source: laerne.github.io)

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- What is life like on this surface?

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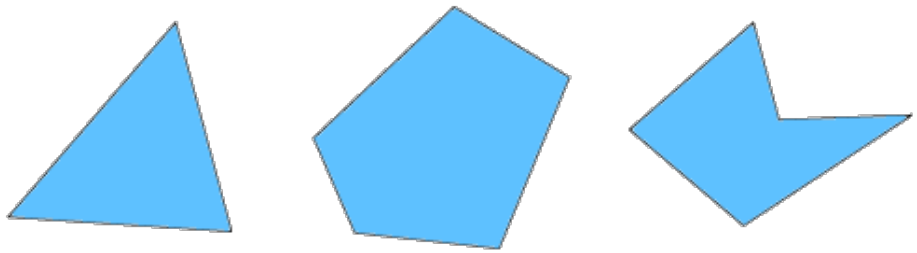
- The inhabitants of Flatland are polygons and line segments.
- They have concepts of North, South, East, and West, but not Up and Down.



Some Flatlanders.

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- What was the object that visited the square?

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- On the other hand, the ball is **solid**, and is a 3-dimensional object.

# MRI

This is how MRIs work!

Can you guess these MRI scans of fruit/vegetables?

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A MRI scan of an onion.

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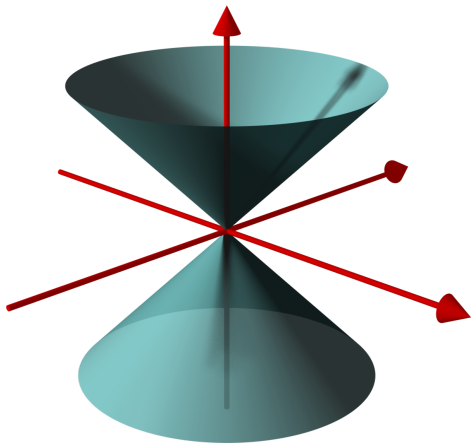
# Cross sections

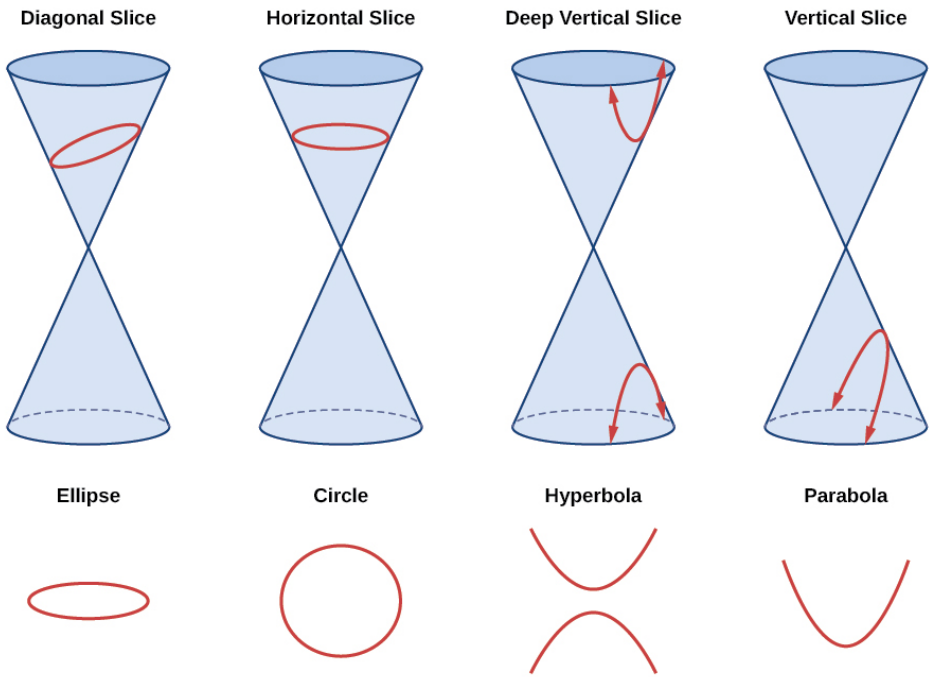
- What kinds of cross-sections do we get if we intersect the surface of a cone with the plane?
- What cross-sections do we get if we intersect a cube with the plane?



A double cone. (Source: Wikipedia)

# Cross sections

1. What kinds of cross-sections do we get if we intersect a double cone with the plane?




This is why these graphs are called **conic sections**. (Source.)

# Cross sections

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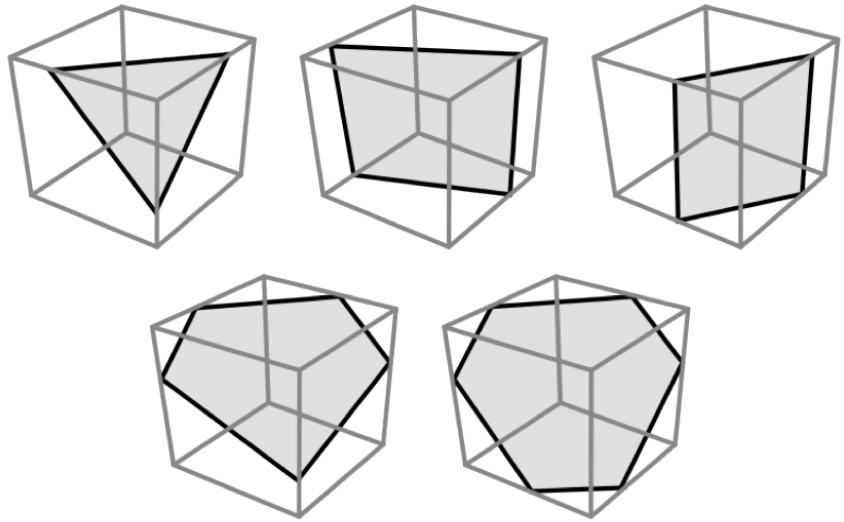

**Source:** Cococubed.com

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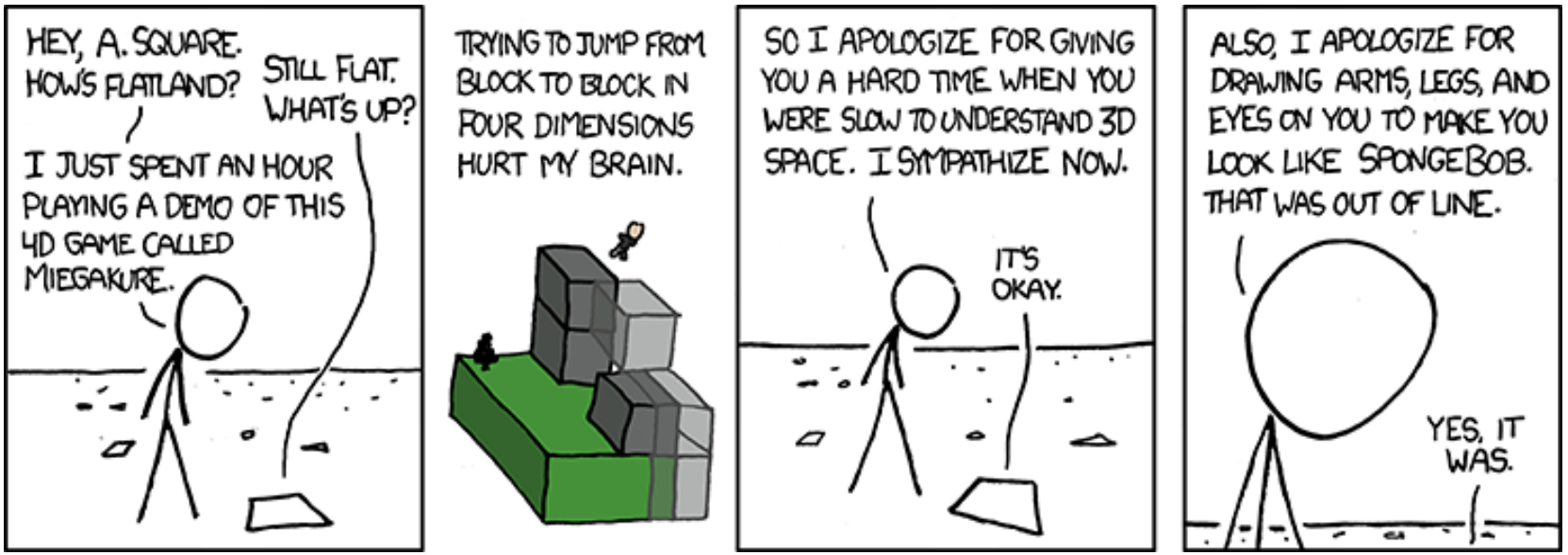
- What kinds of superpowers would a 3D object have in Flatland?
  - A 3D object could see everything in Flatland at once.
  - A 3D object could appear to teleport.

# Hyperspace

What does the story of Flatland mean for us?




Relevant XKCD.

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- Remark: By 4D, we are discussing 4 spatial dimensions. In other words, there is no such thing as "**the** 4th dimension". There is only "**a** 4th dimension."
- Higher dimensions are harder to visualize, so we will often use analogies from Flatland.

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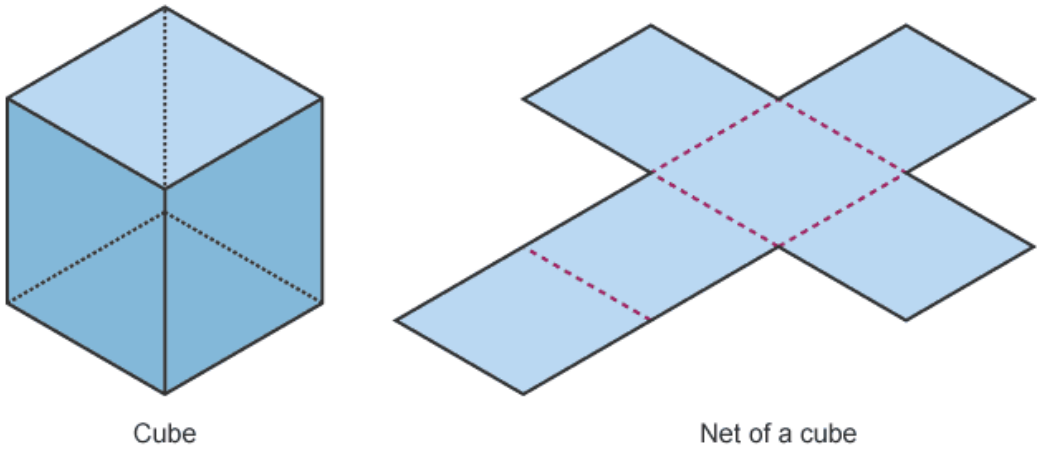
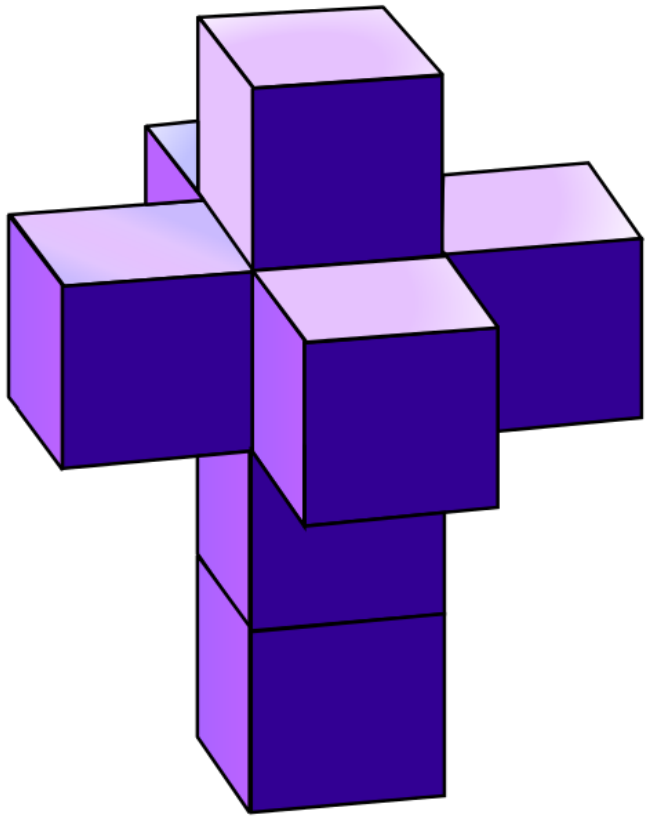



The polyhedral net of a cube.

# Polyhedral Nets



The net of a **tesseract**, aka a hypercube. (Source: Wikipedia)

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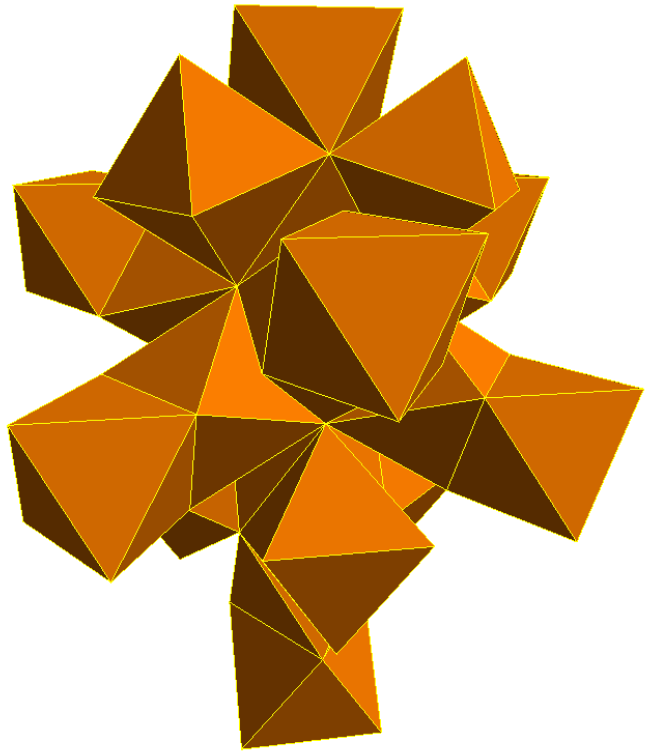

The net of a 24-cell. (Source: Wikipedia)

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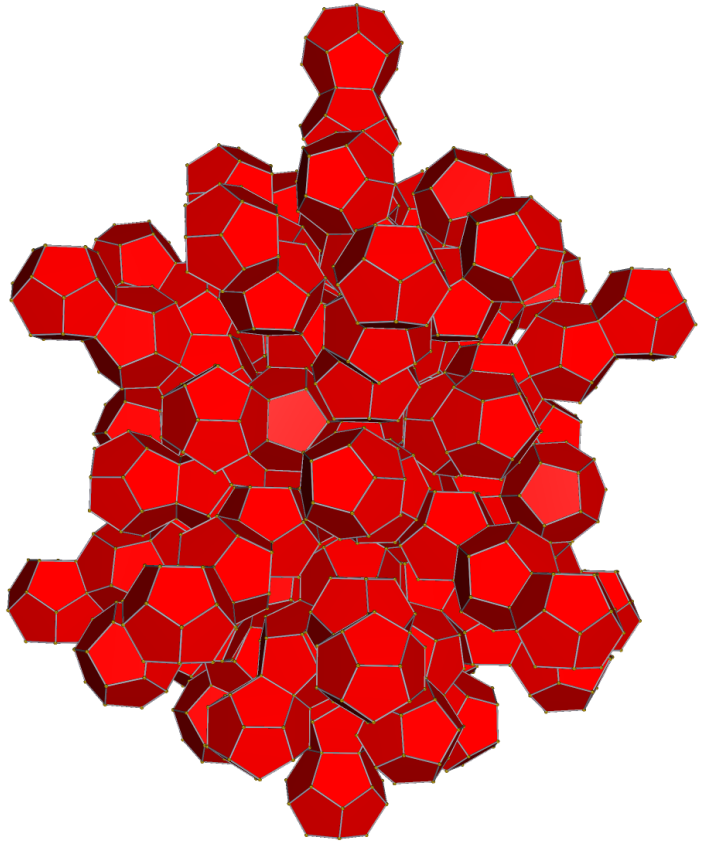

The net of a 120-cell. (Source: Wikipedia)

# Cross Sections

- Here are a few examples of 4D Cross Sections. We will see the tesseract, 24-cell, and the 120-cell. Can you guess which one is which?
- Cross Section 1.
- Cross Section 2.
- Cross Section 3.

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- So another way is to use projections: We can use perspective to draw 3D objects in two dimensions.
- Similarly, we can use perspective to draw 4D objects in 3 dimensions.
- Rotating tesseract.
- Rotating 24-cell.
- Rotating 120-cell.