Spectral Sequence Training Montage, Day 2

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Summer Minicourses 2020

Slides, exercises, and video recordings can be found at
https://web.ma.utexas.edu/SMC/2020/Resources.html

For today, let G be a discrete group.

Remark

Various things in this talk will make sense for compact Lie groups. Figure out which ones!

Is it possible to construct a space X such that such that

$$\pi_n(X) \cong \begin{cases} G & n = 1 \\ 0 & else \end{cases}$$

Yes! Recall covering space theory builds a space X such that $\pi_1(X) = G$.

We need a simply connected space Y such that G acts freely on Y. Then $Y \xrightarrow{p} X = Y/G$ is a universal covering space, and we have

$$G \cong \pi_1(X)/p_*(\pi_1(Y)) \cong \pi_1(X)$$

We can generalize the proof to build a space $BG = K(G, 1)$ such that $\pi_n(BG) \cong \begin{cases} G & n = 1 \\ 0 & else \end{cases}$

We just need a contractible space Y such that G acts freely on Y.

Construction

One construction is the Milnor construction, which constructs EG, a contractible CW complex such that G acts freely.

$$EG = colim_i G^{*i}$$

Then $BG = EG/G$, the quotient space of the G-action.

We then obtain a fibration

$$G \to EG \to BG$$

Example

$E\mathbb{Z} = \mathbb{R}$, $B\mathbb{Z} \simeq S^1$

Example

$E\mathbb{Z}/2 = S^\infty$, $B\mathbb{Z}/2 \simeq \mathbb{R}P^\infty$

Example

$B(G \times H) \simeq BG \times BH$

Definition

The group cohomology of G is defined to be the cohomology of BG:

$$H^*(G;\mathbb{Z}) := H^*(BG;\mathbb{Z})$$

More generally, given a $\mathbb{Z}$-module M, one can define

Definition

The group cohomology of G with coefficients in M is defined to be the cohomology of BG:

$$H^*(G;M) := H^*(BG;M)$$

Example

$$H^i(\mathbb{Z}/2;\mathbb{Z}) := H^i(\mathbb{R}P^\infty;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i \text{ odd } \geq 1 \\ \mathbb{Z}/2 & i \text{ even } \geq 2 \end{cases}$$

Example

$H^i(\mathbb{Z}/2;\mathbb{F}_2) := H^i(\mathbb{R}P^\infty;\mathbb{F}_2) \cong \mathbb{F}_2[x]$, with $|x| = 1$

Proposition

If $|G|$ is invertible in R, $H^n(G;R) = 0$ for $n \geq 1$.

Proposition

If N is a normal subgroup of G, there is a Serre fibration of classifying spaces

$$BN \to BG \to BG/N$$

Example

Given a G-module M, the spectral sequence associated to the above fibration is the Lyndon-Hochschild-Serre spectral sequence:

$$E_2^{s,t} = H^s(G/N; H^t(N; M)) \Rightarrow H^{s+t}(G; M)$$

Remark

For a Serre fibrations $Y \to X \to BG$, observe that we are no longer in the situation where $\pi_1(B)$ is trivial / acts trivially on $H^*(F)$.

We must consider (group) cohomology with local coefficients. This takes into account the action of $\pi_1(BG) = G$ on M.

A $\mathbb{Z}$-module with G-action is the same as a $\mathbb{Z}G$-module. So when considering the category of modules with G action, one can instead consider the category of $\mathbb{Z}G$-modules.

One defines the cohomology of BG with local coefficients M to be

$$H^*(G; M) := H^*(\mathrm{Hom}_{\mathbb{Z}G}(C_n(EG), M))$$

Example

Suppose we have a universal cover $\tilde{X} \to X$ with $\pi_1(X) = G$. Then we have a Serre fibration $\tilde{X} \to X \to BG$.

We will show that there is an isomorphism

$$H^*(X; \mathbb{Q}) \to (H^*(\tilde{X}; \mathbb{Q}))^G$$

We have the LHS spectral sequence

$$E_2^{s,t} = H^s(G; H^t(\tilde{X}; \mathbb{Q})) \Rightarrow H^{p+q}(X; \mathbb{Q})$$

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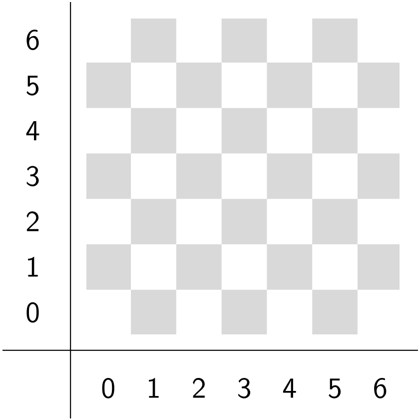


We will now switch gears and discuss the homotopy fixed-point spectral sequence.

Let X be a spectrum with G action. One can construct homotopy fixed point spectum

$$X^{hG} = F((EG)_+, X)^G$$

Example

Let X be a spectrum with naïve (Borel) G-action. That is, $X \in \mathrm{Fun}(BG, \mathrm{Sp})$. Then

$$\mathrm{Res}_e^G(X)^{hG} = X^G$$

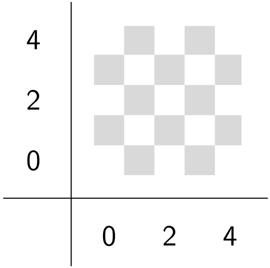
Theorem

We have the homotopy fixed point spectral sequence, which takes in input the spectrum R with a G-action, and computes $\pi_*(R^{hG})$:

$$E_2^{s,t}(R) = H^s(G; \pi_t(R)) \Rightarrow \pi_{t-s}(R^{hG})$$

and differentials

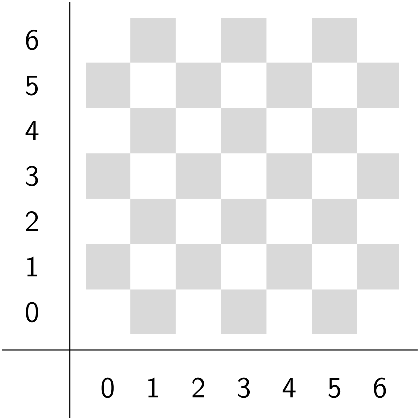
$$d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$$



Example

Let G be a finite group and M a G-module. This induces a G-action on the Eilenberg-Maclane spectrum HM. Then we have

$$\pi_*(HM^{hG}) \cong H^{-*}(G; M)$$


6
5
4
3
2
1
0
0 1 2 3 4 5 6

Example

Let KU denote the ring spectrum that represents complex topological K-theory. We know that

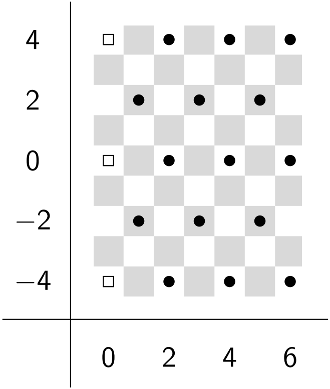
$$\pi_*(KU) = \mathbb{Z}[x^{\pm 1}]$$

Complex conjugation on complex vector bundles induces a $\mathbb{Z}/2$ action on KU. We can then form the homotopy fixed points $KU^{h\mathbb{Z}/2}$.

Let KO denote the spectrum representing real topological K-theory. We claim that

$$KO \simeq KU^{h\mathbb{Z}/2}$$

$$E_2^{s,t} = H^s(\mathbb{Z}/2; \pi_t(KU)) \Rightarrow \pi_{t-s}(KU^{h\mathbb{Z}/2})$$



The $\mathbb{Z}/2$-HFPSS computing $\pi_*(KU^{h\mathbb{Z}/2}) \cong \pi_*(KO)$. $\square = \mathbb{Z}$, $\bullet = \mathbb{Z}/2$.

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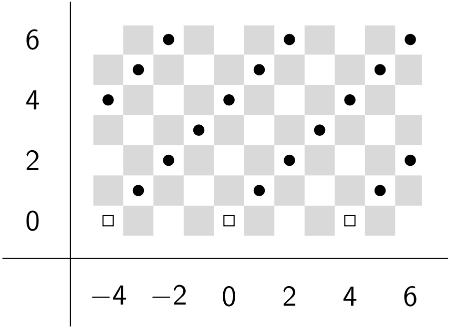

The Adams graded $\mathbb{Z}/2$-HFPSS computing $\pi_*(KU^{h\mathbb{Z}/2}) \cong \pi_*(KO)$.
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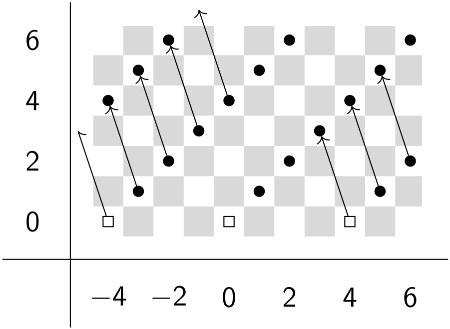

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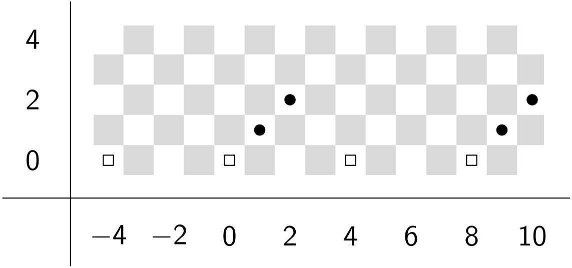

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Example

Consider the short exact sequence of groups $N \to G \to G/N$. This gives a fiber sequence of spaces

$$G/N \to BN \to BG$$

If we take cochains with Hk-valued coefficients, we have a morphism of ring spectra

$$k^{hG} \to k^{hN}$$

and an action of G/N on k^{hN} such that $k^{hG} \simeq (k^{hN})^{hG/N}$.

Exercise: Compare this HFPSS with the Serre spectal sequence associated to $BN \to BG \to BG/N$.

Example

Let G be a finite group and $E \to B$ be a principal G-bundle. For a fixed prime p, we can form the cochain algebras $R = F(E_+, H\mathbb{F}_p)$ and $S = F(B_+, H\mathbb{F}_p)$. Then

$$R \simeq S^{hG}$$

Observe that we have a fibration $E \to B \to BG$, and in fact this exhibits B as the homotopy orbits of E:

$$B \simeq E_{hG} := (EG \times_G E)$$

by comparing this to the **Borel fibration** $E \to E_{hG} \to BG$.

Problem Session

You can find the exercises at
https://web.ma.utexas.edu/SMC/2020/Resources.html.

We are using the free (sign-up required) A Web Whiteboard website. The link will be posted in the chat, as well as on the slack channel.

Future problem sessions will be from 1-1:30pm and 2:30-3pm CDT.